Certain properties of the solutions to evolutionary equations

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Contents

1 Evolutionary equations and C^0 semigroups

A number of problems originating in applications fall into the class of a semilinear evolutionary equations of the form

$$\dot{u} + Au = F(t, u), \ t > 0, \tag{1}$$

in a Banach space X, where A is a linear main part of the equation and F is an appropriately defined nonlinear operator. Having been given any 'concrete' problem we make a suitable choice of the space X and of the maps A, F, so that the properties of the considered system can be satisfactorily displayed. D. Henry's monograph (1981) brings much of these ideas as well as it comes up with a bunch of essential examples.

It happens very often that the 'vector field' F does not depend on t; i.e. the equation is autonomous. If this is the case it becomes natural to write the solution in time t in the form $S(t)u_0$, where $u_0 = S(0)u_0$ is an initial condition that is satisfied by the solution at the initial moment $t = 0$. Whenever the solution exists for all $t \geq 0$ and is unique we obtain this way a family of (usually nonlinear) operators $S(t) : X \to X$, $t \geq 0$, satisfying the conditions

$$S(0) = Id, \ S(t_1 + t_2) = S(t_1) + S(t_2), \ t_1, t_2 \geq 0. \tag{2}$$

Reasonable assumptions on A and F provide the continuous dependence of the solutions on t and u_0 so that the map

$$[0, \infty) \times X \ni (t, u_0) \to S(t)u_0 \in X, \tag{3}$$

is continuous. Then we say that the family $\{S(t), t \geq 0\}$ is a C^0-semigroup in X.

Let $S(t)u_0$ be certain solution. We say that $v \in X$ is a limit point of this solution if there is a sequence $t_n \to \infty$ such that $S(t_n)u_0 \to v$. The set consisting of all the limit points of the solution $S(t)u_0$ is denoted by $\omega(u_0)$. This is so called ω-limit set of u_0;

$$\omega(u_0) = \bigcap_{t \geq 0} cl_X \bigcup_{s \geq t} S(s)u_0.$$

On a plane the ω-limit set can e.g. correspond to a motion whose path goes close to a circumference (like $\dot{r} = -(r - r_0)$, $\dot{\phi} = 1$ in polar coordinates). In a simpler example of a one dimensional o.d.e. $\dot{y} = -\sigma y$, the ω-limit set is easily seen to be trivial. Its only element is zero which is simultaneously (Lyapunov) stable equilibrium; in fact (globally) asymptotically stable.

This concept, if generalized, leads to the notion of a global attractor. This is to catch the global behavior of the solutions and come up with the relevant information about the limit state of the system. Following J. K. Hale's monograph (1988) we say that a (nonempty) set $\boldsymbol{A} \subset X$ is a global attractor for $\{S(t)\}$ iff $\boldsymbol{A}$ is compact, invariant and attracts bounded subsets of X; i.e. $S(t)B$ tends to $\boldsymbol{A}$ in the sense of Hausdorff semidistance in X. A consequence of the definition is that the global attractor, if it exists, is unique and is a 'minimal' set in the class of compact invariant sets attracting bounded subsets of X. Mention should be made that compactness is essential here. The example

$$\dot{x} = -\varepsilon x, \ \dot{y} = -y, \tag{4}$$

with $\varepsilon = 0$ shows that there exists an invariant set attracting bounded sets on the plane (ox axis). We can say that this is a minimal global attractor following O. A. Ladyzenskaya (1987,1991). If $\varepsilon > 0$, then there exist a (compact) global attractor $\{(0,0)\}$ which can be immediately related with the appearance of dissipativeness along ox axis. The notion of dissipativeness is defined through the existence of a bounded subset of a phase space attracting points and comes back to N. Levinson (1944). The following situation is then typical for the semigroups that are compact ($cl_X S(t)B$ is compact whenever $t > 0$ and B is bounded in X).

Theorem 1 *(J. E. Billotti and J. P. LaSalle (1971); J. K. Hale and G. Raugel (1991)) A compact and point dissipative C^0 semigroup $\{S(t)\}$ possesses a (compact) global attractor $\boldsymbol{A}$.*

Example 1 *When the semigroup of global solutions $\{S(t)\}$ is governed by the system of ordinary differential equations*

$$\{\dot{x}_i = f_i(x_1, \ldots, x_N), \ i = 1, \ldots, N, \tag{5}$$

where $\boldsymbol{f} = (f_1, \ldots, f_N)$ is a $C^1(\mathbb{R}^N, \mathbb{R}^N)$ vector field, a global attractor exists if and only if $\{S(t)\}$ is dissipative ($S(t)$ is a compact map for each $t \geq 0$ since the closed unit ball is compact in $\mathbb{R}^N$).

An interesting feature of abstract parabolic equations (autonomous equations of type (1) with A being a sectorial operator) is that the analogous result can be formulated even though the space of initial data is infinite dimensional. This happens when the resolvent of A is compact.

Example 2 *When the semigroup of global solutions $\{S(t)\}$ is governed by* (1) *with a sectorial operator having compact resolvent in a Banach space X, then a global attractor exists if and only if $\{S(t)\}$ is dissipative (maps $S(t)$ are then compact whenever $t > 0$).*

If we deal with a one dimensional o.d.e. with $C^1(\mathbb{R}, \mathbb{R})$-nonlinearity it is easy to give a necessary and sufficient condition guaranteeing that there exists a semigroup of global solutions enjoying dissipativeness property; this is

$$sf(s) < 0, \ |s| > s_0 > 0. \tag{6}$$

In general this is a cumbersome task (not to say impossible).

Example 3 *(fluid mechanics) Navier-Stokes system*

$$\begin{cases} u_t = \nu \Delta u - \nabla p - (u, \nabla)u + h, \ t > 0, \ x \in \Omega, \\ div\, u = 0, \ t > 0, \ x \in \Omega, \\ u = 0, \ t > 0, \ x \in \partial\Omega, \\ u(0,x) = u_0(x), \ x \in \Omega \end{cases} \tag{7}$$

can be viewed as an abstract parabolic equation with a sectorial operator having compact resolvent (the existence of smooth solutions in three dimensional case is unknown).

Example 4 *(solid fuel ignition model) Frank-Kamenetskii equation*

$$\begin{cases} u_t = \Delta u + \lambda e^u, \ x \in \Omega, \ t > 0, \\ u = 0, \ x \in \partial\Omega, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \Omega \end{cases} \tag{8}$$

possesses solutions that cease to exist in a final time (H. Fujita (1969)).

It seems challenging try to say something about (possible) stability in the large for the problems like these. Before, let us nevertheless turn our attention towards the behavior of solutions that - looking from the point of view of the above discussion - could be called 'atypical'.

2 Atypical properties of solutions

Three phenomena (of specifically nonlinear character) should be discussed: blow up, quenching and extinction. The past three decades have witnessed a growing interest in these phenomena which has been displayed through a number of relevant publications. Following B. Kawohl (1992) it should be mentioned that - in the context of parabolic equations - diffusion becomes somehow neglected and behavior of the solution is in certain degree controlled by an ordinary differential equation (thus it seems reasonable to start the discussion from o.d.e).

A simple observation is that the solution usually will not exist for all $t \geq 0$ if the right hand side grows faster than linearly.

Example 5 *We may picture a solution to* $\dot{y} = y^2$*,* $y(0) = 1$ *(see Fig. 1). The maximal interval of existence is* $(-\infty, 1)$*.*

In the light of the above example a similar behavior of partial differential equations can be observed.

Example 6 *If* $p = 2$ *in the following Neumann type initial boundary value problem*

$$\begin{cases} u_t = \Delta u + u^p,\ t > 0,\ x \in \Omega,\ p > 1, \\ \frac{\partial u}{\partial N} = 0 \quad na\ \partial\Omega, \\ u(0,x) = u_0(x),\ x \in \Omega. \end{cases} \tag{9}$$

then the solution of the previous o.d.e is spatially independent solution to (9)*.*

Needless to say that this can be generalized to the higher order problems. Instead, following T. Dlotko (1989), let us show that

Proposition 1 *Each smooth, nonnegative (*$\not\equiv 0$*) solution to* (9) *becomes unbounded in a final time.*

Proof. Local solutions exist and are nonnegative (we take into account sufficiently smooth initial data). Then

$$\int_\Omega u_t dx = \int_\Omega \Delta u dx + \int_\Omega u^p dx,$$

and since the normal derivative is zero on the boundary the divergence theorem and Höldera inequality ensure that

$$\frac{d}{dt}\int_\Omega u dx = \int_{\partial\Omega} \nabla u \cdot dS + \int_\Omega u^p dx \geq c_\Omega \left(\int_\Omega u dx\right)^p \ (c_\Omega = |\Omega|^{1-p}).$$

Therefore the spatial average $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u dx$ satisfies the differential inequality

$$\dot{y} \geq y^p \quad (p > 1). \tag{10}$$

This average becomes thus unbounded in finite time (c.f. Fig. 1 for $p = 2$). The mean value theorem implies that the solution becomes unbounded in a final time as well. ∎

It may not be easy to answer the questions when, where and how the blow up occurs. Actually it is not that bad in the case of (9).

Proposition 2 *A life time τ_{u_0} of any sufficiently smooth solution corresponding to a positive initial condition u_0 fulfils the estimate*

$$T \le \frac{1}{(p-1)\bar{u}_0^{p-1}}. \tag{11}$$

Proof. We use the previous argument to get

$$\frac{1}{p-1}\left(-\frac{1}{y^{p-1}}+\frac{1}{y_0^{p-1}}\right) \ge t,$$
$$y^{p-1} \ge \frac{1}{\frac{1}{y_0^{p-1}}-(p-1)t},$$

and

$$\sup_{\Omega}|u| \ge \frac{1}{|\Omega|}\int_{\Omega} u dx = \bar{u} \ge \left[\frac{1}{\bar{u}_0^{p-1}}-(p-1)t\right]^{-\frac{1}{p-1}},$$

from which (11) follows. ∎

For the questions 'where' and 'how' e.g. references like A. Friedman and B. McLeod (1985) or F. B. Weissler (1984) can be consulted. A solution can be infinite in certain points (for a one dimensional problem like (9) this can be a single point) and stay finite in the other ones. Also, when blow up occurs in time τ, the solution can satisfy the relation $s^{\frac{1}{p-1}}u(x,\tau-s) \to const$ as $s \searrow 0$.

Of course unbounded may be the derivative of the solution.

Example 7 *('gradient catastrophe'; Fig. 2) Coming back to J. Smoller's monograph (1983) we may picture the derivative u_x of a smooth solution to the Burgers equation*

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,\ t>0, x\in\mathbb{R}, \tag{12}$$

The opposite to blow up is extinction (dead core) although in the analysis of this phenomenon similar technique is applicable. If we go back to the work of A. Friedman and M. Herrero (1987) (or earlier to A. S. Kalashnikov (1974)), we may find the examples of problems in which the solutions corresponding to positive initial data become zero in a final time.

Example 8 *Picture the solution to $\dot{y} = -\lambda y^q$, $y(0) = y_0 > 0$, in case when $q \in (0,1)$ and $\lambda > 0$ (see Fig. 3).*

Example 9 *(equation with absorption) Check that*

$$u_{T_0}(x,t) = \begin{cases} [\lambda(1-q)(T_0-t)]^{\frac{1}{1-q}}, \ t < T_0, \\ 0, \ t \geq 0. \end{cases} \tag{13}$$

solves

$$u_t = \Delta u - \lambda u^q, \ t > 0, \ x \in \mathbb{R}^N, \ q \in (0,1), \ \lambda > 0. \tag{14}$$

Quenching seems to be a more sophisticated phenomenon.

Example 10 *Following H. Kawarada (1975) picture the solution to* $\dot{y} = \frac{1}{1-y}$, $y(0) = 0$ *(see Fig. 4). It disappears in* 1 *(its time derivative becomes unbounded when* $t \nearrow 0$ *whereas the solution approaches* 1*).*

Example 11 *Similar phenomenon can be observed for*

$$u_t = \Delta u + \frac{1}{1-u}, \ t > 0, \ x \in \Omega, \tag{15}$$

equipped with

$$u(0,x) = 0, \ x \in \Omega, \tag{16}$$

and the homogeneous Neumann boundary condition since the spatially independent solution is given as in Fig. 4.

Coming back to H. Kawarada (1975) and considering

$$u(t,0) = u(t,l) = 0, \ t > 0, \ \text{(one dimensional case)}, \tag{17}$$

let us show that

Theorem 2 *In case of* (15)-(17) *the quenching accurs if* $l > 2\sqrt{2}$.

Proof. It is crucial to note that the relation $l > 2\sqrt{2}$ makes the solution to go in a finite time arbitrarily close to 1.

Suppose that this is not the case; i.e. the solution u is defined for all $t \geq 0$. Compare it with the solution v to

$$\begin{cases} v_t = v_{xx} + 1, \ t > 0, x \in (0,l), \\ v(t,0) = v(t,l) = 0, \ t > 0, \\ v(0,x) = 0, \ x \in (0,l). \end{cases} \tag{18}$$

As a consequence $u \geq v$ (since $\frac{1}{1-u} \geq 1$ whenever $1 > u \geq 0$). However, the solution to (18) is globally defined and tends as $t \to \infty$ to a (unique) equilibrium (see Fig. 5).

Since the stationary solution to (18) is given explicitly fo $x \in (0, l)$ as $v_l(x) = \frac{1}{2}x(l-x)$, we infer that $v_l\left(\frac{l}{2}\right) = \frac{l^2}{8} > 1$ provided $l > 2\sqrt{2}$. In the latter case v tending to v_l has to reach 1 in a final time. This nevertheless is not possible unless we allow that u can reach 1 in a final time (which we did not). ∎

J. K. Hale's monograph (1988) shows how to look at quenching from the point of view of dynamical systems and how the phase portrait of the system changes with respect to l.

3 Abstract continuation result

It is not that we are not able to give an equivalent condition for the existence of the semigroup of global solutions with orbits of bounded sets bounded. Theorem below, which comes back to J. W. Cholewa and T. Dlotko (2000) brings such a result.

Assumption 1 *Let X be a Banach space, $A : D(A) \to X$ a sectorial and positive operator in X and let $F : X^\alpha \to X$ be Lipschitz continuous on bounded subsets of X^α for some $\alpha \in [0, 1)$.*

Consider the Cauchy problem

$$\dot{u} + Au = F(u),\ t > 0,\ u(0) = u_0, \tag{19}$$

and the two conditions:

(A_1) Relation $S(t)u_0 = u(t, u_0)$ defines on X^α, corresponding to (1), a C^0 semigroup $\{S(t)\}$ of global X^α solutions having orbits of bounded sets bounded.

(A_2) It is possible to choose

- *a Banach space Y, with $D(A) \subset Y$,*
- *a locally bounded function $c : [0, +\infty) \longrightarrow [0, +\infty)$,*
- *a nondecreasing function $g : [0, +\infty) \longrightarrow [0, +\infty)$,*
- *a certain number $\theta \in [0, 1)$,*

such that, for each $u_0 \in X^\alpha$, both conditions below hold

$$\|u(t, u_0)\|_Y \le c(\|u_0\|_{X^\alpha}),\ t \in (0, \tau_{u_0}), \tag{20}$$

$$\|F(u(t, u_0))\|_X \le g(\|u(t, u_0)\|_Y)(1 + \|u(t, u_0)\|^\theta_{X^\alpha}),\ t \in (0, \tau_{u_0}), \tag{21}$$

Theorem 3 *Under Assumption 1, conditions (A_1) and (A_2) are equivalent.*

It becomes clear that the question concerning possibility of global in time continuation of solutions to semilinear parabolic equations is completely solved when the nonlinear term is subordinated to an α-power ($\alpha \in [0, 1)$) of the operator appearing in the linear main part of the equation. This turns us towards the concept of 'growth limitation'. We face twice the necessity of limiting the growth of the nonlinearity with respect to u. The first time is in the context of local solutions when we want to ensure Lipschitz continuity of Assumption 1. The second time is to get the global well posedness result. The growth limitations appearing in both of the mentioned questions have been frequently studied and through the past two decades some progress has been achieved; the results of H. Amann (1985), W. von Wahl (1985), A. N. Carvalho and J. W. Cholewa (2005) and the notion of an *ε-regular solution* introduced by J. M . Arrieta and A. N. Carvalho (2000) can be mentioned here.

In case of problems introduced in Example 3 or Example 4 the result based on the subordination technique does not help. The reason lies in the critical growth of the corresponding nonlinear term. Nevertheless, following F. Rothe (1987), we are still able to use the 'dynamical system' approach to describe the behavior of the mentioned systems in the large.

4 Asymptotics of solutions to parabolic equations with possible blow-up

For any $h \in [L^r(\Omega)]^n$ the system (7) (where $n \geq 2$, $\nu > 0$ is a viscosity constant and $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary $\partial\Omega$ of class C^{2^+}) can be viewed as an abstract parabolic problem

$$u_t + A_r u = F_r u + P_r h,\ t > 0,\ u_{|t=0} = u_0. \qquad (22)$$

where $A_r = -\nu P_r \Delta$ considered with the domain $D(A_r) = X_r \cap \{\phi \in [W^{2,r}(\Omega)]^n : \phi_{|\partial\Omega} = 0\}$ is sectorial in $X_r = cl_{[L^r(\Omega)]^n}\{\phi \in [C_0^\infty(\Omega)]^n : div\, \phi = 0\}$, and $F_r u = -P_r(u, \nabla)u$. Here X_r is a subspace of the divergence free vector fields on which the Navier-Stokes system is 'projected' with the aid of P_r (see D. Fujiwara, H. Morimoto, 1977); also Y. Giga (1985) for the characterization of fractional power spaces X_r^α).

If $\alpha \in [\frac{1}{2}, 1)$ and $r > n$, then F_r takes X_r^α into X_r and is Lipschitz on bounded sets. Hence we have that:

Proposition 3 *The problem* (22) *is locally well posed in* X_r^α *(*$\alpha \in [\frac{1}{2}, 1)$*,* $r > n \geq 2$*) and*

$$u(\cdot, u_0) \in C([0, \tau_{u_0}), X_r^\alpha) \cap C^1((0, \tau_{u_0}), X_r^{1^-}) \cap C((0, \tau_{u_0}), X_r^1), \tag{23}$$

where $[0, \tau_{u_0})$ *denotes the maximal interval of existence of the solution corresponding to* $u_0 \in X_r^\alpha$*.*

Example 12 *('order of magnitude') Relation*

$$\|F_r w\|_{X_r} \leq c_r \|w\|^2_{[W^{1,r}(\Omega)]^n}, \ w \in [W^{1,r}(\Omega)]^n, \ r > n \geq 2,$$

(see Y. Giga and T. Miyakawa (1985)) can be used to show that

$$\|F_r(u(t, u_0)) + h\|_{X_r} \leq C\left(\|u(t, u_0)\|_{X_r}, \|h\|_{X_r}\right)\left(1 + \|u(t, u_0)\|_{X_r^1}\right), \tag{24}$$

as long as the solution exists. From this it can be seen that the nonlinearity in the Navier-Stokes system has the same 'order of magnitude' as the linear main part relatively to an $[L^r(\Omega)]^n$ *estimate of solution with* $r > n$*. It should be emphasized that such an estimate is however generally unknown.*

It is important to notice (see e.g. J. W. Cholewa and T. Dlotko (2003)) that the Navier-Stokes system can enjoy the Lyapunov functional.

Lemma 1 *Fix* $\alpha \in [\frac{1}{2}, 1)$*,* $r > n \geq 2$*,* $u_0 \in X_r^\alpha$*. Suppose that there exists a stationary solution* $u_S \in X_r^1$ *to* (22) *such that*

$$\|u_S\|_{[W^{1,\infty}(\Omega)]^n} < \frac{\nu}{C_\Omega^2}, \tag{25}$$

(C_Ω *being a constant from the Poincaré inequality) and define*

$$\mathcal{L}(\phi) = \|\phi - u_S\|_{L^2(\Omega)]^N}, \ \phi \in X_r^\alpha.$$

If $u(\cdot, u_0)$ *is a solution to* (22)*, then* $\mathcal{L}(u(\cdot, u_0))$ *is non increasing as long as* $u(\cdot, u_0)$ *exists. Furthermore, if* $\mathcal{L}(u(\cdot, u_0)) \equiv const$ *in the interval of existence of* $u(\cdot, u_0)$*, then* $u \equiv u_S$*.*

Proof. The equation for $v = u - u_S$ reads

$$v_t = \nu P_r \Delta v - P_r(v, \nabla)v - P_r(v, \nabla)u_S - P_r(u_S, \nabla)v. \tag{26}$$

Also

$$P_r v = P_2 v \ \text{ for } \ v \in [L^r(\Omega)]^n, \ r \geq 2;$$

and P_2 is a selfadjoint bounded operator on $[L^2(\Omega)]^n$ (see D. Fujiwara, H. Morimoto (1977)). Thus, for $\phi, \psi, \eta \in X_r^1$,

$$\langle P_r(\phi, \nabla)\psi, \eta\rangle_{[L^2(\Omega)]^n} = \langle P_2(\phi, \nabla)\psi, \eta\rangle_{[L^2(\Omega)]^n} = \langle(\phi, \nabla)\psi, \eta\rangle_{[L^2(\Omega)]^n}$$

and

$$\langle P_r\Delta\eta, \eta\rangle_{[L^2(\Omega)]^n} = \langle\Delta\eta, \eta\rangle_{[L^2(\Omega)]^n}.$$

Therefore, multiplying both sides of (26) in $[L^2(\Omega)]^n$ by v (belonging to the class described in (23)), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2_{[L^2(\Omega)]^n} &\leq -\nu\sum_{i=1}^{n}\|\nabla v_i\|^2_{L^2(\Omega)} + \|u_S\|_{[W^{1,\infty}(\Omega)]^n}\sum_{i=1}^{n}\|v_i\|^2_{L^2(\Omega)} \\ &\leq \left(-\frac{\nu}{C_\Omega^2} + \|u_S\|_{[W^{1,\infty}(\Omega)]^n}\right)\|v\|^2_{[L^2(\Omega)]^n} \leq 0.\end{aligned} \tag{27}$$

On the other hand, if we have

$$\mathcal{L}(u(t, u_0)) := \frac{1}{2}\|u(t, u_0) - u_S\|^2_{[L^2(\Omega)]^n} \equiv const. \tag{28}$$

where $u = u(\cdot, u_0)$ is a solution to (22), then $v = u - u_S$ solves (26), which now leads to the relation:

$$0 = \frac{d}{dt}[\mathcal{L}(u(t, u_0))] = \langle v_t, v\rangle_{[L^2(\Omega)]^n} \leq \left(-\frac{\nu}{C_\Omega^2} + \|u_S\|_{[W^{1,\infty}(\Omega)]^n}\right)\|v\|^2_{[L^2(\Omega)]^n} \leq 0.$$

From above inequality $\|v\|_{[L^2(\Omega)]^n} \equiv 0$ and, consequently, $u \equiv u_S$. ∎

Following consideration of J. W. Cholewa and T. Dlotko (2003) for $h \in [L^r(\Omega)]^n$, $r > n \geq 2$ and $\alpha \in [\frac{1}{2}, 1)$ let

$$\mathcal{U}_{h,r,\alpha} := \{u_0 \in X_r^\alpha : \sup_{t\in[0,\infty)} \|u(t, u_0)\|_{X_r^\alpha} < \infty\}$$

denote the set of all globally bounded in time X_r^α solutions to (22).

Lemma 2 *Suppose that* (22) *possesses a stationary solution u_S for which* (25) *holds and let $u_0 \in U_{h,r,\alpha}$. Then, the solution $u(\cdot, u_0)$ to* (22) *converges in X_r^α to u_S.*

Proof. If $u_0 \in U_{h,r,\alpha}$ then $\tau_{u_0} = \infty$ and $\gamma^+(u_0)$ is bounded in X_r^α. Thus $cl_{X_r^\alpha}\gamma^+(u_0)$ is also bounded and is contained in $U_{h,r,\alpha}$. Since the resolvent operators $(\lambda I - A_r)^{-1}$ are compact for $\lambda \in \rho(A_r)$, this implies further compactness of $cl_{X_r^\alpha}\gamma^+(u(1+t, u_0))$ in X_r^α. Lemma 1 implies now that $\omega(u_0) = \{u_S\}$, which completes the proof. ∎

Global behavior of orbits of the Navier-Stokes system now follows.

Corollary 1 *If* (22) *possesses a stationary solution* u_S *for which* (25) *holds, then* $\{u_S\} \subset U_{h,\alpha,r}$ *is a maximal compact invariant subset of* X_r^α *and any solution* $u(\cdot, u_0)$ *of* (22) *either blows-up in* X_r^α *in finite or infinite time or converges to* $\{u_S\}$.

In particular, there exists $\mu_0 > 0$ *such that* u_S *is asymptotically stable whenever* $\|h\|_{[L^r(\Omega)]^n} < \mu_0$.

If in addition $n = 2$, *and* $h \equiv 0$, *then zero is a globally asymptotically stable equilibrium.*

Proof. The alternative is an immediate consequence of Lemma 2. Next, if $\|h\|_{[L^r(\Omega)]^n}$ is sufficiently small (see J. W. Cholewa and T. Dlotko (1998)) for detail calculations of an appropriate upper bound), there exists a compact invariant set $\mathcal{A}$ attracting certain neighborhood of zero. In the light of our previous considerations we conclude that $\mathcal{A}$ lies in $U_{h,\alpha,r}$ and coincides with $\{u_S\}$. Indeed, if we take any complete invariant orbit contained in $\mathcal{A}$, then we observe that the Lyapunov functional defined by $\mathcal{L}(w) := \frac{1}{2}\|w - u_S\|^2_{[L^2(\Omega)]^n}$, $w \in U_{h,\alpha,r}$, must be constant along this orbit. Otherwise among elements of $U_{h,\alpha,r}$ there would be two different equilibria, which is excluded by Lemma 1.

In particular, if $h \equiv 0$, then $\mathcal{U}_{h,r,\alpha} = X_r^\alpha$ for $n = 2$, and the compact global attractor, which is known to exist for the Navier-Stokes system in this case, is a single point set $\{0\}$. ∎

The exponential reaction-diffusion problem (8) can be viewed as the Cauchy problem (19) in the Hilbert space $L^2(\Omega)$ (here A is a negative Laplacian with the domain $H^2(\Omega) \cap H_0^1(\Omega)$). In the one dimensional case ($\Omega = (-1, 1)$) the local well posedness in $H_0^1(\Omega)$ is immediate and

$$L(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx - \lambda \int_\Omega e^\phi dx \tag{29}$$

is a Lyapunov function on a metric space V consisting of all $H_0^1(\Omega)$-initial conditions u_0 for which the corresponding solution stays globally bounded in $H_0^1(\Omega)$. If we consider $\lambda \in (0, \lambda^*)$ (where $\lambda^* \sim 0.878$ as in the monograph by J. Bebernes and D. Eberly (1989)), then there are exactly two equilibria $w^\pm$ and, as shown in H. Fujita (1969), unless the initial condition u_0 lies below the maximal (stationary w^-) solution the corresponding solution $u(\cdot, u_0)$ to (8) blows up in a finite time. The semigroup on V, if restricted to a complete metric space $\{\phi \in H_0^1(\Omega) : w^+ \leq \phi \leq w^-\}$ is compact and point dissipative. Therefore, it possesses a compact global attractor. Since it is connected, the equilibria w^+, w^- are connected by a heteroclinic orbit, which in fact is unique (see P. Brunovský and B. Fiedler (1986)). This is sufficient to conclude (see R. Czaja (2004)) that any solution to (8) either blows up in a finite or infinite time, or - if it stays bounded - approaches a maximal compact invariant set.

5 References

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6 Appendix

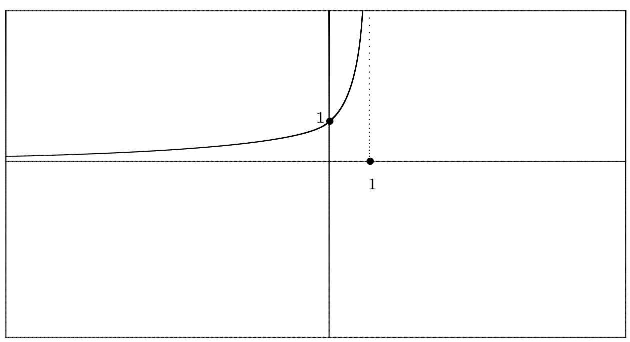

Figure 1: The solution to $\dot{y} = y^2$, $y(0) = 1$.

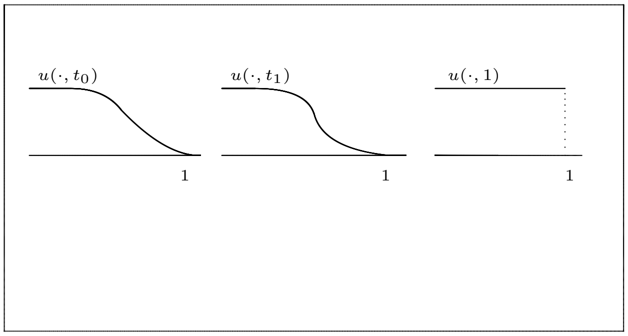

Figure 2: Intuitively the wave will have to 'break'.

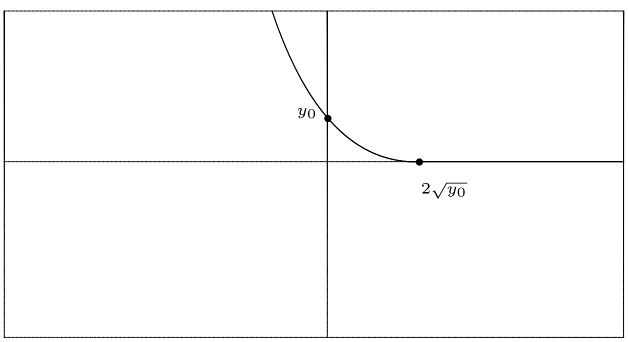

Figure 3: Solution $y = \left(\sqrt{y}_0 - \frac{1}{2}t\right)^2$, $t \in (-\infty, 2\sqrt{y_0}]$, $y = 0$, $t > 2\sqrt{y_0}$, to $\dot{y} = -\sqrt{y}$, $y(0) = y_0 > 0$.

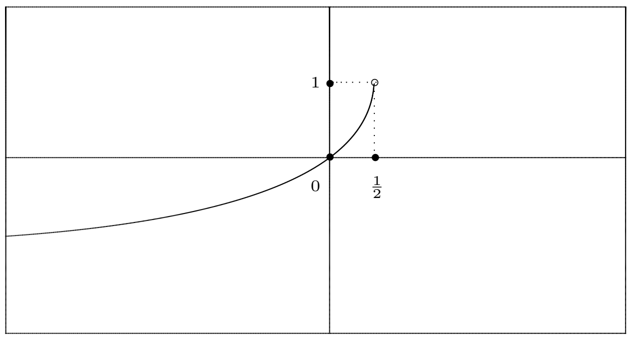

Figure 4: The solution to $\dot{y} = \frac{1}{1-y}$, $y(0) = 0$ on the maximal interval of existence.

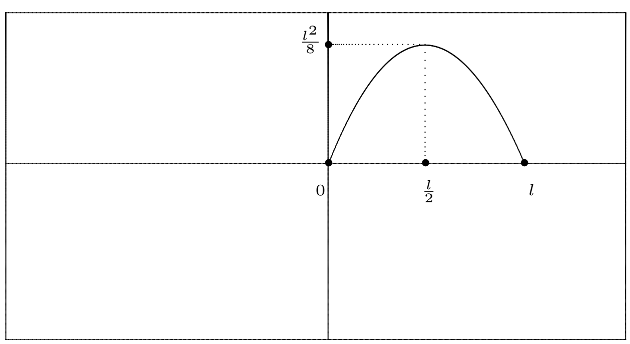

Figure 5: Equilibrium of (18).