# Sistemas Dinâmicos Não Lineares
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# Exponential Splitting and Dichotomy

## Definição (Exponential Splitting)

*A linear evolution process* $\{L(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ *has* **exponential splitting**, *with constant* $M \geqslant 1$, *exponents* $\gamma, \rho \in \mathbb{R}$, *with* $\gamma > \rho$, *and a family of projections* $\{Q(t) : \ t \in \mathbb{R}\} \subset \mathcal{L}(X)$, *if*

- *i)* $Q(t)L(t,\tau) = L(t,\tau)Q(\tau)$, *for all* $t \geqslant \tau$,
- *ii)* $L(t,\tau) : Im(Q(\tau)) \to Im(Q(t))$ *is an isomorphism, with inverse denoted by* $L(\tau,t)$,
- *iii)* *the following estimates hold*

$$\begin{aligned} \|L(t,\tau)Q(\tau)\|_{\mathcal{L}(X)} &\leqslant Me^{-\rho(t-\tau)}, \quad t \leqslant \tau, \\ \|L(t,\tau)(I - Q(\tau))\|_{\mathcal{L}(X)} &\leqslant Me^{-\gamma(t-\tau)}, \quad t \geqslant \tau. \end{aligned} \tag{1}$$

# Inertial Manifolds

Consider the following semilinear differential initial value problem

$$\begin{aligned} &\dot{u} = A(t)u + f(t,u), \;\; t > \tau, \\ &u(\tau) = u_0 \in X, \end{aligned} \tag{2}$$

with $f : \mathbb{R} \times X \to X$ continuous, $f(t,0) = 0$, for all $t \in \mathbb{R}$ and uniformly Lipschitz in the second variable with Lipschitz constant $\ell > 0$, i.e., $\|f(t,u) - f(t,\tilde{u})\| \leqslant \ell \|u - \tilde{u}\|$ for any $(t,u),(t,\tilde{u}) \in \mathbb{R} \times X$.

Assume that the family of linear operators $\{A(t) : t \in \mathbb{R}\}$ (not bounded) defines a linear evolution process $\{L(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$, i.e., for each $(\tau, u_0) \in \mathbb{R} \times X$, the 'solution' of the linear problem,

$$\begin{aligned} \dot{u} &= A(t)u, \quad t \geqslant \tau, \\ u(\tau) &= u_0 \in X, \end{aligned} \tag{3}$$

is given by $u(t,\tau,u_0) = L(t,\tau)u_0$, for $t \geqslant \tau$, $L(t,t) = Id_X$, $L(t,s)L(s,\tau) = L(t,\tau)$, $t \geqslant s \geqslant \tau$ and $[\tau, \infty) \ni t \mapsto L(t,\tau)u_0 \in X$ is continuous, for all $(\tau, u_0) \in \mathbb{R} \times X$.

With this, solutions of (2) define a nonlinear evolution process $\{T(t,\tau) : t \geqslant \tau\} \subset \mathcal{C}(X)$ given by the variation of constants formula, that is,

$$T(t,\tau)u = L(t,\tau)u + \int_\tau^t L(t,s) f(s, T(s,\tau)u)\, ds, \quad t \geqslant \tau, u \in X. \tag{4}$$

## Teorema

*Suppose that the linear evolution process* $\{L(t,\tau): t \geqslant \tau\}$ *has exponential splitting, with constant* $M \geqslant 1$*, exponents* $\gamma > \rho$ *and a family of projections* $\{Q(t): t \in \mathbb{R}\}$*. If* $f: \mathbb{R} \times X \to X$ *is continuous,* $f(t,0) = 0$*,* $f(t,\cdot): X \to X$ *is Lipschitz continuous with Lipschitz constant* $\ell > 0$*, for all* $t \in \mathbb{R}$*, and*

$$\frac{\gamma - \rho}{\ell} > \max\{M^2 + 2M + \sqrt{8M^3}, 3M^2 + 2M\}, \tag{5}$$

*then there is a continuous function*

$$\begin{aligned} \Sigma^* : \mathbb{R} \times X &\to X \\ (t,u) &\mapsto \Sigma^*(t,u) \end{aligned} \tag{6}$$

*such that* $\Sigma^*(t,u) = \Sigma^*(t, Q(t)u) = (I - Q(t))\Sigma^*(t,u)$ *and* $\Sigma^*(t,0) = 0$*, for all* $t \in \mathbb{R}$*.*

*In addition* $\Sigma^*(t,\cdot) : X \to X$ *Lipschitz continuous with Lipschitz constant* $\kappa = \kappa(\gamma, \rho, \ell, M) > 0$*, for all* $t \in \mathbb{R}$*, that is,*
$\|\Sigma^*(t,u) - \Sigma^*(t,\tilde{u})\| \leqslant \kappa \|u - \tilde{u}\|$*, for all* $(t,u), (t,\tilde{u}) \in \mathbb{R} \times X$*.*

*Moreover, the graph of* $\Sigma^*(t,.)$*, for each* $t \in \mathbb{R}$*, given by*

$$\mathcal{M}(t) := \{u \in X : u = q + \Sigma^*(t,q), q \in Im(Q(t))\}, \qquad (7)$$

*yields an invariant manifold* $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *for the evolution process* $\{T(t,\tau) : t \geqslant \tau\}$ *given by* (4)*.*

*In other words, it is invariant and if*

$$P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u), \ (t,u) \in \mathbb{R} \times X$$

*is the nonlinear projection onto* $\mathcal{M}(t)$*.*

(i) $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *has controlled growth: for* $(\tau, u) \in \mathbb{R} \times X$, $t \leqslant \tau$,

$$\|T(t,\tau)P_{\Sigma^*}(\tau)u\| \leqslant M(1+\kappa)e^{-(\rho+\ell M(1+\kappa))(t-\tau)}\|P_{\Sigma^*}(\tau)u\|. \quad (8)$$

(ii) $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *satisfies: for any* $(\tau, u) \in \mathbb{R} \times X$ *and* $t \geqslant \tau$,

$$\|T(t,\tau)u - P_{\Sigma^*}(t)T(t,\tau)u\| \leqslant M\|(I - P_{\Sigma^*}(\tau))u\|e^{-\delta(t-\tau)}, \quad (9)$$

*where* $\delta := \gamma - M\ell - \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-\ell M(1+\kappa)}$. *If* $\delta > 0$, $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *is an inertial manifold.*

# Stable Manifold of an Invariant Manifold

## Teorema

*Suppose that the linear evolution process* $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential splitting, with constant* $M \geqslant 1$*, exponents* $\gamma > \rho$ *and a family of projections* $\{Q(t) : \ t \in \mathbb{R}\}$*.*
*If* $(\gamma - \rho)/\ell$ *satisfies* (5)*, then there is a continuous function*

$$\begin{aligned} \Theta^* : \mathbb{R} \times X &\to X \\ (t,u) &\mapsto \Theta^*(t,u), \end{aligned} \tag{10}$$

*such that* $\Theta^*(t,u) = \Theta^*(t,(I - Q(t))u) = Q(t)\Theta^*(t,u)$*, and* $\Theta^*(t,0) = 0$ *for all* $t \in \mathbb{R}$*, which is Lipschitz with constant* $\kappa = \kappa(\gamma,\rho,\ell,M) > 0$ *, i.e.,* $\|\Theta^*(t,u) - \Theta^*(t,\tilde{u})\| \leqslant \kappa\|u - \tilde{u}\|$ *for all* $(t,u),(t,\tilde{u}) \in \mathbb{R} \times X$*.*

*Moreover, if* $P_{\Theta^*}(t)u := \Theta^*(t,(I-Q(t))u) + (I-Q(t))u$*, for all* $(t,u) \in \mathbb{R}\times X$*, the family given by*

$$\{Im(P_{\Theta^*}(t)) : t\in\mathbb{R}\} := \{\{P_{\Theta}^*(t,u) : u\in X\} : t\in\mathbb{R}\}, \qquad (11)$$

*is positively invariant such that*

$$\|T(t,\tau)P_{\Theta^*}(\tau)u\| \leqslant M(1+\kappa)e^{-(\gamma - M\ell(1+\kappa))(t-\tau)}\|P_{\Theta^*}(\tau)u\|, \quad (12)$$

$t \geqslant \tau,\ u\in X$*, and*

$$\|u - P_{\Theta^*}(\tau)u\| \leqslant Me^{\hat{\delta}(t-\tau)}\|(I-P_{\Theta^*}(t))T(t,\tau)u\|, \qquad (13)$$

$t\geqslant\tau, u\in X$*, where* $\hat{\delta} = \rho + M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}$.
*Furthermore, if* $\gamma - M\ell(1+\kappa) > 0$*,* $\{Im(P_{\Theta^*}(t)) : t\in\mathbb{R}\}$ *is the stable manifold of the inertial manifold* $\{Im(P_{\Sigma^*}(t)) : t\in\mathbb{R}\}$.

**Proof:** Given $\kappa > 0$ consider the complete metric space

$$\begin{aligned}\mathcal{LB}_{\Theta}(\kappa) = \Big\{ &\Theta \in C(\mathbb{R} \times X, X) : \|\Theta(t,u) - \Theta(t,\tilde{u})\| \leqslant \kappa \|u - \tilde{u}\|, \Theta(t,0) = 0, \\ &\Theta(t,u) = \Theta(t,(I - Q(t))u) \in Im(Q(t)),\ \forall (t,u),(t,\tilde{u}) \in \mathbb{R} \times X \Big\}.\end{aligned} \tag{14}$$

with the metric $\|\Theta - \tilde{\Theta}\| = \sup_{t\in\mathbb{R}} \sup_{u\neq 0} \dfrac{\|\Theta(t,u) - \tilde{\Theta}(t,u)\|}{\|u\|}$.

Next we outline the heuristic procedure that will establish the way of proving that the invariant manifold is given as a graph of a map in $\mathcal{LB}_\Theta(\kappa)$. We are looking for $\Theta \in \mathcal{LB}_\Theta(\kappa)$ with the property that, if $(\tau, \eta) \in \mathbb{R} \times X$, then a solution $u$ of (2), with initial data $u(\tau) = \Theta(\tau, (I - Q(\tau))\eta) + (I - Q(\tau))\eta \in X$, can be decomposed as $u(t) = q(t) + p(t)$, where $q(t) = \Theta(t, p(t))$ for all $t \geqslant \tau$. Thus, $q$ and $p$ must satisfy, for $t \geqslant \tau$,

$$\begin{aligned} q(t) &= L(t,\tau)Q(\tau)\eta + \int_\tau^t L(t,s)Q(s)f(s, p(s) + \Theta(s, p(s)))ds, \\ p(t) &= L(t,\tau)(I - Q(\tau))\eta + \int_\tau^t L(t,s)(I - Q(s))f(s, p(s) + \Theta(s, p(s)))ds. \end{aligned} \tag{15}$$

It follows that

$$\begin{aligned}\|p(t)\| &\leqslant \|L(t,\tau)(I-Q(\tau))\eta\| \\ &\quad + \int_\tau^t \|L(t,s)(I-Q(s))f(s,p(s)+\Theta(s,p(s)))\|ds \\ &\leqslant Me^{-\gamma(t-\tau)}\|\eta\| + \int_\tau^t M\ell e^{-\gamma(t-s)}(1+\kappa)\|p(s)\|ds.\end{aligned}$$

Using Grownwall's inequality,

$$\|p(t)\| \leqslant Me^{-(\gamma-M\ell(1+\kappa))(t-\tau)}\|\eta\|.$$

From this and from the fact that $q(t)=\Theta(p(t))$, we conclude that

$$\begin{aligned}\|L(\tau,t)Q(t)q(t)\| &= \|L(\tau,t)\Theta(p(t))\| \\ &\leqslant \kappa M^2 e^{-(\gamma-\rho-M\ell(1+\kappa))(t-\tau)}\|\eta\|.\end{aligned}$$

Applying $L(\tau,t)Q(t)$ to (15), using that $\Theta(\tau,(I-Q(\tau))\eta)=Q(\tau)\eta$ and making $t\to\infty$ we have

$$0=\Theta(\tau,(I-Q(\tau))\eta)+\int_\tau^\infty L(\tau,s)Q(s)f(s,p(s)+\Theta(s,p(s)))ds.$$

Inspired by this we define the operator $\tilde{G}:\mathcal{LB}_\Theta(\kappa)\to\mathcal{LB}_\Theta(\kappa)$ by

$$\tilde{G}(\Theta)(\tau,\eta)=-\int_\tau^\infty L(\tau,s)Q(s)f(s,p(s)+\Theta(s,y(s)))ds, \quad (16)$$

$(\tau,\eta)\in\mathbb{R}\times X$.

The fact that $\tilde{G}$ is a well-defined contraction is similar to Theorem 1, and we refrain from giving a proof. Hence $\tilde{G}$ admits a unique fixed point $\Theta^* \in \mathcal{LB}_\Theta(\kappa)$ satisfying the desired properties.
We now embark in the proof of (13). For any $(\tau, \eta) \in X$ and $t \geqslant \tau$.

$$p(t) = L(t,\tau)(I - Q(\tau))\eta + \int_\tau^t L(t,s)(I - Q(s))f(s, q(s) + p(s))ds$$

and thus we wish to bound the variable $\eta(t) := T(t,\tau)u - P_{\Theta^*}(t)T(t,\tau)u$ for any $u \in X$ and $t \geqslant \tau$. Note that $\eta(t) = q(t) - \Theta^*(t, p(t))$ due to the definitions in (15).

Define $p^*(s,t)$, for $s \geqslant t$, as

$$\begin{aligned} p^*(s,t) := & L(s,t)p(t) \\ & + \int_t^s L(s,r)(I - Q(r))f(r, \Theta^*(r, p^*(r,t)) + p^*(r,t))dr. \end{aligned} \tag{17}$$

Since $f, \Theta^*$ are Lipschitz with respective constants $\ell, \kappa > 0$, we obtain

$$\begin{aligned} & \|p^*(s,t) - p(s)\| \\ & \leqslant M \int_t^s e^{-\gamma(s-r)} \|f(r, \Theta^*(r, p^*(r,t)) + p^*(r,t)) - f(r, q(r) + p(r))\| dr \\ & \leqslant M\ell \int_t^s e^{-\gamma(s-r)} (\|q(r) - \Theta^*(r, p(r))\| + (1+\kappa)\|p^*(r,t) - p(r)\|) dr, \end{aligned} \tag{18}$$

and, by Grönwall's Lemma,

$$\|p^*(s,t) - p(s)\| \leqslant M\ell \int_t^s e^{-(\gamma - M\ell(1+\kappa))(s-r)} \|\eta(r)\| dr. \tag{19}$$

Also, for $s \geqslant t \geqslant \tau$, we obtain

$$\|p^*(s,\tau) - p^*(s,t)\| \leqslant \|L(s,t)(I-Q(t))[p^*(t,\tau) - p(t)]\|$$
$$+\|\int_t^s L(s,r)(I-Q(r))[f(r,\Theta^*(r,p^*v(r,\tau))+p^*(r,\tau))-f(r,\Theta^*(r,p^*(r,t))+p^*(r,t))]\|dr$$
$$\leqslant M^2\ell e^{-\gamma(s-t)}\int_\tau^t e^{-(\gamma-M\ell(1+\kappa))(t-r)}\|\eta(r)\|dr$$
$$+ M\ell(1+\kappa)\int_t^s e^{-\gamma(s-r)}\|p^*(r,\tau)-p^*(r,t)\|dr,$$

and again by Grönwall's Lemma,

$$\|p^*(s,\tau) - p^*(s,t)\| \leqslant M^2\ell \int_\tau^t e^{-(\gamma-M\ell(1+\kappa))(s-r)}\|\eta(r)\|dr. \quad (20)$$

Now, we use these inequalities to estimate $\|\eta(\tau)\|$. Note that

$$\eta(\tau)-L(\tau,t)Q(t)\eta(t)=q(\tau)-L(\tau,t)q(t)-\Theta^*(\tau,p(\tau))+L(\tau,t)\Theta^*(t,p(t))$$

$$=\int_t^\tau L(\tau,s)Q(s)[f(s,q(s)+p(s))-f(s,\Theta^*(s,p^*(s,\tau))+p^*(s,\tau))]ds$$

$$+\int_t^\infty L(\tau,s)Q(s)[f(s,\Theta^*(s,p^*(s,\tau))+p^*(s,\tau))-f(s,\Theta^*(s,p^*(s,t))+p^*(s,t))]ds.$$

Thus, using (19) and (20), we obtain

$$
\begin{aligned}
&\|\eta(\tau) - L(\tau,t)Q(t)\eta(t)\| \\
&\leqslant M\ell \int_\tau^t e^{-\rho(\tau-s)} \left(\|q(s) - \Theta^*(s,p^*(s,\tau))\| + \|p(s) - p^*(s,\tau)\|\right) ds \\
&+ M\ell(1+\kappa) \int_t^\infty e^{-\rho(\tau-s)} \|p^*(s,t) - p^*(s,\tau)\| ds \\
&\leqslant M\ell \int_\tau^t e^{-\rho(\tau-s)} \|\eta(s)\| ds \\
&+ M^2\ell^2(1+\kappa) \int_\tau^t e^{-(\gamma-\rho-M\ell(1+\kappa))(s-\tau)} \int_\tau^s e^{-(\gamma-M\ell(1+\kappa))(\tau-r)} \|\eta(r)\| dr ds \\
&+ M^3\ell^2(1+\kappa) \int_t^\infty e^{-(\gamma-\rho-M\ell(1+\kappa))(s-\tau)} \int_\tau^t e^{-(\gamma-M\ell(1+\kappa))(\tau-r)} \|\eta(r)\| dr ds.
\end{aligned}
$$

Hence

$$\|\eta(\tau)-L(\tau,t)Q(t)\eta(t)\| \leqslant M\ell\int_\tau^t e^{-\rho(\tau-s)}\|\eta(s)\|ds$$
$$+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\int_\tau^t e^{-(\gamma-\rho-M\ell(1+\kappa))(\tau-r)}e^{-\rho(\tau-r)}\|\eta(r)\|\,dr$$

and we have that

$$\|\eta(\tau)-L(\tau,t)Q(t)\eta(t)\| \leqslant \left[M\ell+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\right]\int_\tau^t e^{-\rho(\tau-r)}\|\eta(r)\|dr.$$

Thus,

$$\|\eta(\tau)\|\leqslant Me^{-\rho(\tau-t)}\|\eta(t)\|+\left[M\ell+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\right]\int_\tau^t e^{-\rho(\tau-r)}\|\eta(r)\|dr.$$

By Grönwall's Lemma, we obtain the bound in (13). $\square$

# The saddle point property

We now obtain the saddle point property as an immediate consequence of Theorems 1 and 2. We define the **unstable** and **stable sets** of a hyperbolic global solution $u_*$ of (2) as

$$W^u(u_*) := \left\{ (\tau, u_0) \in \mathbb{R} \times X : \begin{array}{l} \text{there is a solution } u : (-\infty, \tau] \to X \\ \text{such that } u(\tau) = u_0 \text{ and} \\ \lim_{t \to -\infty} \|u(t) - u_*(t)\|_X = 0 \end{array} \right\} \tag{21a}$$

$$W^s(u_*) := \left\{ (\tau, u_0) \in \mathbb{R} \times X : \begin{array}{l} \text{there is a solution } u : [\tau, \infty) \to X \\ \text{such that } u(\tau) = u_0 \text{ and} \\ \lim_{t \to +\infty} \|u(t) - u_*(t)\|_X = 0 \end{array} \right\} \tag{21b}$$

## Corolário

*Suppose that the linear evolution process $\{L(t,\tau) : t \geqslant \tau\}$ has exponential dichotomy, with constant $M \geqslant 1$, exponent $\gamma > 0$ and a family of projections $\{Q(t) : \ t \in \mathbb{R}\}$.*
*Suppose that $\ell > 0$ is sufficiently small, then there are continuous functions $\Sigma^u \in \mathcal{L}_\Sigma(\kappa)$ and $\Theta^s \in \mathcal{L}_\Theta(\kappa)$ such that the unstable and stable manifolds of $u_* = 0$ are given by*

$$W^u(0) = \{(\tau, u) \in \mathbb{R} \times X : u = Q(\tau)u + \Sigma^u(\tau, Q(\tau)u)\}, \tag{22a}$$
$$W^s(0) = \{(\tau, u) \in \mathbb{R} \times X : u = \Theta^s(\tau, (I - Q(\tau))u) + (I - Q(\tau))u\}. \tag{22b}$$

*Moreover, solutions within the unstable (resp. stable) manifold exponentially decay to zero backwards (resp. forwards) in time, according to (8) and (12).*

**Proof:** For $\ell > 0$ sufficiently small, the condition (5) is satisfied and $\delta > 0$, and thus we obtain the graph of $\Sigma^*$ from Theorem 1. We now prove that the unstable set $W^u(0)$ defined in (21a) coincides with the graph of $\Sigma^u := \Sigma^*$. On one hand, the graph of $\Sigma^u$ is contained in the unstable set by (8). On the other hand, any solution $z : (-\infty, t] \to X$ which backwards converges to zero satisfies, from (9),

$$\begin{aligned}\|z(t) - P_{\Sigma^*}(t)z(t)\| &= \|(I - Q(t))z(t) - \Sigma^u(t, Q(t)z(t))\| \\ &\leqslant M\|(I - P_{\Sigma^*}(\tau))z(\tau)\|e^{-\delta(t-\tau)}, \quad t \geqslant \tau.\end{aligned}$$

Since $\delta > 0$, we obtain that $(I - Q(t))z(t) = \Sigma^u(t, Q(t)z(t))$ for all $t \in \mathbb{R}$ as $\tau \to -\infty$, and thus any element in the unstable set lies in the graph of $\Sigma^u$. The case of stable manifold is analogous applying Theorem 2. □

# Roughness of Exponential Dichotomy

We now prove that the roughness of exponential dichotomy, i.e., that exponential dichotomies are preserved under perturbations. Assume that the linear evolution process $\{L(t,\tau) : t \geqslant \tau\}$ associated to the problem

$$\dot{u} = A(t)u, \;\; t \geqslant \tau, \quad u(\tau) = u_0. \tag{23}$$

has exponential dichotomy with constant $M$ and exponent $\gamma > 0$ and consider the linear evolution process $\{T(t,\tau) : t \geqslant \tau\}$, associated to a perturbation of it, given by the linear equation,

$$\dot{u} = A(t)u + B(t)u, \;\; t \geqslant \tau, \quad u(\tau) = u_0. \tag{24}$$

where the map $t \mapsto B(t) \in \mathcal{L}(X)$ is strongly continuous for $t \in \mathbb{R}$ and $\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant \ell$, for some suitably small $\ell > 0$.

Recall that, as in (4), the evolution process $\{T(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ associated to (24) is given by

$$T(t,\tau) = L(t,\tau) + \int_\tau^t L(t,s)B(s)T(s,\tau)\,ds, \quad t \geqslant \tau. \tag{25}$$

We wish to prove that (24) has exponential dichotomy for suitably small $\ell$.

This result can be obtained by firstly applying Theorems 1 and 2 in a linear setting, which are suitable in order to establish the existence of the linear invariant manifold and its stable manifold (see Corollary 2) and then apply it to (24) with $\gamma > 0$ and $\rho = -\gamma$.

## Corolário

*If* $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential splitting with constant* $M$*, exponents* $\gamma > \rho$ *and projections* $\{Q(t) : \ t \in \mathbb{R}\}$ *and* (5) *is satisfied, then*

- *There are maps* $\Sigma^*, \Theta^* : \mathbb{R} \times X \to X$, $\Sigma^*(t,\cdot), \Theta^*(t,\cdot) \in \mathcal{L}(X)$ *and* $\|\Sigma^*(t,u)\| \leqslant \kappa \|u\|_X$, $\|\Theta^*(t,u)\|_X \leqslant \kappa \|u\|_X$ *for all* $(t,u) \in \mathbb{R} \times X$ *and for some* $\kappa = \kappa_\ell > 0$;
- *The graph* $\mathcal{G}(\Sigma^*)$ *of* $\Sigma^*$ *is an invariant family and* (9) *holds, the graph* $\mathcal{G}(\Theta^*)$ *of* $\Theta^*$ *is a positively invariant family;*
- *The evolution process* $\{T(t,\tau) : t \geqslant \tau\}$ *given by* (25) *satisfies*
$$\begin{aligned} \|T(t,\tau)P_{\Sigma^*}(\tau)\| &\leqslant M(1+\kappa)e^{-(\rho+M\ell(1+\kappa))(t-\tau)}, \quad t \leqslant \tau, \\ \|T(t,\tau)P_{\Theta^*}(\tau)\| &\leqslant M(1+\kappa)e^{-(\gamma-M\ell(1+\kappa))(t-\tau)}, \quad t \geqslant \tau, \end{aligned} \tag{26}$$
*where* $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u)$ *and* $P_{\Theta^*}(t)u := \Theta^*(t,(I-Q(t))u) + (I-Q(t))u$, $t \in \mathbb{R}$.

**Proof:** The proof is a direct consequence of Theorems 1 and 2 in the case that $f(t,\cdot)$ is linear and uniformly (with respect to $t$) bounded. Note that, the linearity of $\Sigma(t,\cdot)$ follows since $f(t,\cdot)$ is linear, and thereby $G(\Sigma)$ given is also linear. Consequently, the fixed point, $G(\Sigma^*)(t,u)=\Sigma^*(t,u)$, is linear. Similarly, $\tilde{G}$ in equation (16) is also linear and so is $\Theta^*$. $\square$

Next, we show the robustness of the exponential dichotomy.

## Corolário

*Suppose that* $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential dichotomy with constant* $M \geqslant 1$, *exponent* $\gamma > 0$ *and family of projections* $\{Q(t) : t \in \mathbb{R}\}$. *If* $\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant \ell$, *where* $0 < \ell < \frac{2\gamma}{3M(M+1)}$, *then* $\{T(t,\tau) : t \geqslant \tau\}$ *has exponential dichotomy, that is, there are projections* $\{Q_\ell(t) : t \in \mathbb{R}\}$ *with* $T(t,\tau) : Im(Q_\ell(\tau)) \to Im(Q_\ell(t))$ *being an isomorphism,* $t \geqslant \tau$, *and*

$$\begin{aligned} \|T(t,\tau)Q_\ell(\tau)\|_{\mathcal{L}(X)} &\leqslant M_\ell e^{\gamma_\ell(t-\tau)}, \quad t \leqslant \tau \\ \|T(t,\tau)(I - Q_\ell(\tau))\|_{\mathcal{L}(X)} &\leqslant M_\ell e^{-\gamma_\ell(t-\tau)}, \quad t \geqslant \tau, \end{aligned} \tag{27}$$

*where* $M_\ell := M(1+\kappa_\ell)/(1-2\kappa_\ell) > 1$ *and* $\gamma_\ell := \gamma - \ell M(1+\kappa_\ell) > 0$ *for the Lipschitz constant* $\kappa_\ell$ *obtained in Corollary 2.*
*Moreover,*

$$\sup_{t\in\mathbb{R}} \|Q(t) - Q_\ell(t)\|_{\mathcal{L}(X)} \leqslant \frac{2\kappa_\ell}{1 - 2\kappa_\ell}. \tag{28}$$

**Proof:** If $P_{\Theta^*}(t)u := (I - Q(t))u + \Theta^*(t, (I - Q(t))u)$ and $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u)$, for $(t, u) \in \mathbb{R} \times X$, where $\Sigma^*$ and $\Theta^*$ are the bounded linear maps obtained in Corollary 2, with norm less than $\kappa_\ell > 0$. We will prove that $X = Im(P_{\Sigma^*}(t)) \oplus Im(P_{\Theta^*}(t))$, for every $t \in \mathbb{R}$. That is, we show that, for each $(t, u) \in \mathbb{R} \times X$,

$$\begin{aligned} \mathcal{I}_u(t) &: X \to X \\ & v \mapsto \mathcal{I}_u(t)v := u - \Sigma^*(t, v) - \Theta^*(t, v), \end{aligned} \tag{29}$$

has a unique fixed point. If that is the case, for each $(t, u) \in \mathbb{R} \times X$, there exists a unique $v_u \in X$ such that $\mathcal{I}_u(t)v_u = v_u$, that is,

$$u - \Sigma^*(t, v_u) - \Theta^*(t, v_u) = v_u = Q(t)v_u + (I - Q(t))v_u, \text{ or} \tag{30}$$

$$\begin{aligned} u &= Q(t)v_u + \Sigma^*(t, v_u) + (I - Q(t))v_u + \Theta^*(t, v_u) \\ &= P_{\Sigma^*}(t)v_u + P_{\Theta^*}(t)v_u, \end{aligned} \tag{31}$$

Which is the unique representation of $u$ as a sum of elements of $Im(P_{\Sigma^*}(t))$ and $Im(P_{\Theta^*}(t))$ and proves the desired decomposition.

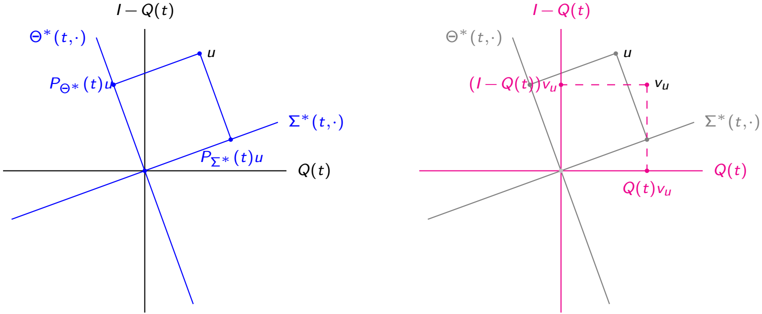


Figure: Given a point $u \in X$, we find a unique point $v_u \in X$ such that $P_{\Sigma^*}(t)u = Q(t)v_u + \Sigma^*(t, Q(t)v_u)$ and $P_{\Theta^*}(t)u = (I - Q(t))v_u + \Theta^*(t, (I - Q(t))v_u)$.

In order to show that $\mathcal{I}_u(t)$ has a unique fixed point, note that $\mathcal{I}_u(t)$ is a contraction on $X$, since

$$\begin{aligned}\|\mathcal{I}_u(t)v-\mathcal{I}_u(t)\tilde{v}\|&=\|\Sigma^*(t,\tilde{v})-\Sigma^*(t,v)+\Theta^*(t,\tilde{v})-\Theta^*(t,v)\|,\\&\leqslant 2\kappa\|v-\tilde{v}\|,\end{aligned} \tag{32}$$

for any $v,\tilde{v}\in X$, as the graphs $\Sigma^*,\Theta^*$ are Lipschitz with constant $\kappa=\kappa_\ell>0$. Thus, $\mathcal{I}_u(t)$ is a contraction for each $(t,u)\in\mathbb{R}\times X$, and for all $\kappa\in[\kappa_-,\min\{1/2,\min\{\kappa_+,\kappa_*\}\})$, since the hypothesis on $\ell$ in Corollary 3 implies that $\kappa_-<1/2$ and thus any $\kappa$ as above implies that we have a contraction.
Note that, for each $u\in X$, since $v_u$ is the unique element of $X$ satisfying $v_u=u-\Sigma^*(t,v_u)-\Theta^*(t,v_u)$, the map $u\mapsto v_u$ is a linear bounded operator such that

$$\|v_u\|_X\leqslant\frac{\|u\|_X}{1-2\kappa}. \tag{33}$$

For each $t \in \mathbb{R}$, define $Q_\ell(t) \in \mathcal{L}(X)$ the linear projection onto $R(P_{\Sigma^*}(t))$ along $R(P_{\Theta^*}(t))$, which can be written as $Q_\ell(t)u := P_{\Sigma^*}(t)v_u$ due to the first part of the proof. Its complementary projection is given by $(I - Q_\ell(t))u = P_{\Theta^*}(t)v_u$, for each $(t, u) \in \mathbb{R} \times X$.
From Corollary 2, we have that $\{R(Q_\ell(t)) : t \in \mathbb{R}\}$ is invariant and $\{R(I - Q_\ell(t)) : t \in \mathbb{R}\}$ is positively invariant. Thus $T(t,\tau)Q_\ell(\tau) = Q_\ell(t)T(t,\tau)$, for every $t \geqslant \tau$. Equations (26) and (33) imply the desired bounds (27). This proves that $\{T(t,\tau) : t \geqslant \tau\}$ has exponential dichotomy with constant $M_\ell := M(1+\kappa)/(1-2\kappa)$ and exponent $\gamma_\ell := \gamma - M\ell(1+\kappa) > 0$.

Lastly, we prove the bound in equation (28), that is, the continuous dependence of the projections $\{Q(t) : t \in \mathbb{R}\}$ and $\{Q_\ell(t) : t \in \mathbb{R}\}$, correponding to the exponential dichotomies of the respective evolution processes $\{L(t,\tau) : t \geqslant \tau\}$ and $\{T(t,\tau) : t \geqslant \tau\}$. Consider $u \in X$, which can be uniquely decomposed as $u = v_u + \Sigma^*(t, v_u) + \Theta^*(t, v_u)$. Hence, $Q(t)u = Q(t)v_u + \Theta^*(t, v_u)$, since $Q(t)\Sigma^*(t, v_u) = 0$, and $Q_\ell(t)u = Q(t)v_u + \Sigma^*(t, v_u)$, by definition of $Q_\ell(t)$ and because $Q_\ell(t)\Theta^*(t, v_u) = 0$. Therefore,

$$Q(t)u - Q_\ell(t)u = \Theta^*(t, v_u) - \Sigma^*(t, v_u). \tag{34}$$

Since the maps $\Sigma^*, \Theta^*$ are Lipschitz with constant $\kappa_\ell > 0$, and due to equation (33), we obtain the desired bound (28). $\square$