Sistemas Dinâmicos Não Lineares
Vigésima Quarta Aula

Alexandre Nolasco de Carvalho

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Exponential Splitting and Dichotomy

Definição (Exponential Splitting)

A linear evolution process $\{L(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ *has* **exponential splitting**, *with constant* $M \geqslant 1$, *exponents* $\gamma, \rho \in \mathbb{R}$, *with* $\gamma > \rho$, *and a family of projections* $\{Q(t) : \ t \in \mathbb{R}\} \subset \mathcal{L}(X)$, *if*

i) $Q(t)L(t,\tau) = L(t,\tau)Q(\tau)$, *for all* $t \geqslant \tau$,

ii) $L(t,\tau) : Im(Q(\tau)) \to Im(Q(t))$ *is an isomorphism, with inverse denoted by* $L(\tau,t)$,

iii) *the following estimates hold*

$$\begin{aligned} \|L(t,\tau)Q(\tau)\|_{\mathcal{L}(X)} &\leqslant Me^{-\rho(t-\tau)}, \quad t \leqslant \tau, \\ \|L(t,\tau)(I - Q(\tau))\|_{\mathcal{L}(X)} &\leqslant Me^{-\gamma(t-\tau)}, \quad t \geqslant \tau. \end{aligned} \tag{1}$$

Inertial Manifolds

Consider the following semilinear differential initial value problem

$$\begin{aligned} &\dot{u} = A(t)u + f(t,u), \ \ t > \tau, \\ &u(\tau) = u_0 \in X, \end{aligned} \tag{2}$$

with $f : \mathbb{R} \times X \to X$ continuous, $f(t,0) = 0$, for all $t \in \mathbb{R}$ and uniformly Lipschitz in the second variable with Lipschitz constant $\ell > 0$, i.e., $\|f(t,u) - f(t,\tilde{u})\| \leqslant \ell \|u - \tilde{u}\|$ for any $(t,u),(t,\tilde{u}) \in \mathbb{R} \times X$.

Assume that the family of linear operators $\{A(t) : t \in \mathbb{R}\}$ (not bounded) defines a linear evolution process $\{L(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$, i.e., for each $(\tau, u_0) \in \mathbb{R} \times X$, the 'solution' of the linear problem,

$$\begin{aligned} \dot{u} &= A(t)u, \quad t \geqslant \tau, \\ u(\tau) &= u_0 \in X, \end{aligned} \tag{3}$$

is given by $u(t,\tau,u_0) = L(t,\tau)u_0$, for $t \geqslant \tau$, $L(t,t) = Id_X$, $L(t,s)L(s,\tau) = L(t,\tau)$, $t \geqslant s \geqslant \tau$ and $[\tau,\infty) \ni t \mapsto L(t,\tau)u_0 \in X$ is continuous, for all $(\tau, u_0) \in \mathbb{R} \times X$.

With this, solutions of (2) define a nonlinear evolution process $\{T(t,\tau) : t \geqslant \tau\} \subset \mathcal{C}(X)$ given by the variation of constants formula, that is,

$$T(t,\tau)u = L(t,\tau)u + \int_\tau^t L(t,s) f(s, T(s,\tau)u)\, ds, \quad t \geqslant \tau, u \in X. \tag{4}$$

Teorema

Suppose that the linear evolution process $\{L(t,\tau): t \geqslant \tau\}$ *has exponential splitting, with constant* $M \geqslant 1$*, exponents* $\gamma > \rho$ *and a family of projections* $\{Q(t): t \in \mathbb{R}\}$*. If* $f: \mathbb{R} \times X \to X$ *is continuous,* $f(t,0) = 0$*,* $f(t,\cdot): X \to X$ *is Lipschitz continuous with Lipschitz constant* $\ell > 0$*, for all* $t \in \mathbb{R}$*, and*

$$\frac{\gamma - \rho}{\ell} > \max\{M^2 + 2M + \sqrt{8M^3}, 3M^2 + 2M\}, \tag{5}$$

then there is a continuous function

$$\begin{aligned} \Sigma^* : \mathbb{R} \times X &\to X \\ (t,u) &\mapsto \Sigma^*(t,u) \end{aligned} \tag{6}$$

such that $\Sigma^*(t,u) = \Sigma^*(t, Q(t)u) = (I - Q(t))\Sigma^*(t,u)$ *and* $\Sigma^*(t,0) = 0$*, for all* $t \in \mathbb{R}$*.*

In addition $\Sigma^*(t,\cdot): X \to X$ *Lipschitz continuous with Lipschitz constant* $\kappa = \kappa(\gamma, \rho, \ell, M) > 0$*, for all* $t \in \mathbb{R}$*, that is,* $\|\Sigma^*(t,u) - \Sigma^*(t,\tilde{u})\| \leqslant \kappa \|u - \tilde{u}\|$*, for all* $(t,u), (t,\tilde{u}) \in \mathbb{R} \times X$*.*

Moreover, the graph of $\Sigma^*(t,.)$, *for each* $t \in \mathbb{R}$, *given by*

$$\mathcal{M}(t) := \{u \in X : u = q + \Sigma^*(t,q), q \in Im(Q(t))\}, \qquad (7)$$

yields an invariant manifold $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *for the evolution process* $\{T(t,\tau) : t \geqslant \tau\}$ *given by* (4).

In other words, it is invariant and if

$$P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u), \ (t,u) \in \mathbb{R} \times X$$

is the nonlinear projection onto $\mathcal{M}(t)$.

(i) $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *has controlled growth: for* $(\tau, u) \in \mathbb{R} \times X$, $t \leqslant \tau$,

$$\|T(t,\tau)P_{\Sigma^*}(\tau)u\| \leqslant M(1+\kappa)e^{-(\rho+\ell M(1+\kappa))(t-\tau)}\|P_{\Sigma^*}(\tau)u\|. \quad (8)$$

(ii) $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *satisfies: for any* $(\tau, u) \in \mathbb{R} \times X$ *and* $t \geqslant \tau$,

$$\|T(t,\tau)u - P_{\Sigma^*}(t)T(t,\tau)u\| \leqslant M\|(I - P_{\Sigma^*}(\tau))u\|e^{-\delta(t-\tau)}, \quad (9)$$

where $\delta := \gamma - M\ell - \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-\ell M(1+\kappa)}$. *If* $\delta > 0$, $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *is an inertial manifold.*

Prova: The proof is divided into two parts. First, we show that there is a function Σ^* yielding the graph of the invariant manifold, as desired. Second, we show that this graph is exponentially dominated.

For the first part, given $\kappa > 0$, consider the following complete metric space,

$$\mathcal{LB}_\Sigma(\kappa) := \Big\{ \Sigma \in C(\mathbb{R}\times X, X) : \sup_{t\in\mathbb{R}} \frac{\|\Sigma(t,u)-\Sigma(t,\tilde{u})\|}{\|u-\tilde{u}\|} \leqslant \kappa, \\ \Sigma(t,0)=0, \Sigma(t,u)=\Sigma(t,Q(t)u) \in N(Q(t), \forall t\in\mathbb{R} \Big\} \tag{10}$$

with the metric $\|\Sigma - \tilde{\Sigma}\| := \sup_{t\in\mathbb{R}} \sup_{u\neq 0} \dfrac{\|\Sigma(t,u) - \tilde{\Sigma}(t,u)\|}{\|u\|}$.

We are looking for $\Sigma \in \mathcal{LB}_\Sigma(\kappa)$ such that, if $(\tau, \eta) \in \mathbb{R} \times X$, then a solution u of (2) with initial data $u(\tau) = Q(\tau)\eta + \Sigma(\tau, Q(\tau)\eta) \in X$ can be decomposed as $u(t) = q(t) + p(t)$, where $p(t) = \Sigma(t, q(t))$ for all $t \in \mathbb{R}$. Thus, q and p must satisfy

$$q(t) = L(t,\tau)Q(\tau)\eta + \int_\tau^t L(t,s)Q(s)f(s, q(s) + \Sigma(s, q(s)))ds, \tag{11a}$$

for $t \leqslant \tau$ and

$$p(\tau) = L(\tau,t)(I - Q(t))p(t) + \int_t^\tau L(\tau,s)(I - Q(s))f(s, q(s) + \Sigma(s, q(s))ds, \tag{11b}$$

for $t \leqslant \tau$.

First, let us control the growth of $q(t)$. Since $\{L(t,s) : t \geqslant s\}$ has exponential splitting, $f(t,0) = 0$ and f and Σ are Lipschitz with respective constants ℓ and κ, we obtain

$$\|q(t)\| \leqslant Me^{-\rho(t-\tau)}\|\eta\| + \int_t^\tau \ell M e^{-\rho(t-s)}(1+\kappa)\|q(s)\| ds, \quad t \leqslant \tau. \tag{12}$$

Then, by Grönwall's Lemma,

$$\|q(t)\| \leqslant Me^{(\rho+M\ell(1+\kappa))(\tau-t)}\|\eta\|, \quad t \leqslant \tau. \tag{13}$$

Heuristically, since we wish that $p(t) = \Sigma(t, q(t))$ for $\Sigma \in \mathcal{LB}_\Sigma(\kappa)$, the growth in equation (**??**) implies that the limit $e^{-\gamma(\tau-t)}\|p(t)\| \to 0$, as $t \to -\infty$.

Thus, due to the exponential splitting of $\{L(t,\tau) : t \geqslant \tau\}$, the first term in (**??**) goes to zero as $t \to -\infty$, yielding

$$p(\tau) = \int_{-\infty}^{\tau} L(\tau, s)(I - Q(s))f(s, q(s) + \Sigma(s, q(s))ds. \qquad (14)$$

Hence, to prove that $\Sigma \in \mathcal{LB}_\Sigma(\kappa)$ that satisfies $p(\tau) = \Sigma(\tau, Q(\tau)\eta)$, it is equivalent to find a fixed point of the following map,

$$G(\Sigma)(\tau, \eta) := \int_{-\infty}^{\tau} L(t, s)(I - Q(s))f(s, q(s) + \Sigma(s, q(s)))ds. \quad (15)$$

Next, we show that $G : \mathcal{LB}_\Sigma(\kappa) \to \mathcal{LB}_\Sigma(\kappa)$ is a well defined contraction in the complete metric space $\mathcal{LB}_\Sigma(\kappa)$.

Let $\eta, \tilde{\eta} \in X$, $\Sigma, \tilde{\Sigma} \in \mathcal{LB}_\Sigma(\kappa)$ with corresponding solutions $q(t), \tilde{q}(t)$ of (**??**).

Thus, for $t \leqslant \tau$,

$$\begin{aligned}
\|q(t) - \tilde{q}(t)\| &\leqslant Me^{\rho(\tau-t)}\|\eta-\tilde{\eta}\| \\
&+ M\int_t^\tau e^{\rho(s-t)}\|f(s,q(s)+\Sigma(s,q(s))) - f(s,\tilde{q}(s)+\tilde{\Sigma}(s,\tilde{q}(s)))\|ds \\
&\leqslant Me^{\rho(\tau-t)}\|\eta-\tilde{\eta}\| \\
&+ \ell M\int_t^\tau e^{-\rho(t-s)}\Big(\|q(s)-\tilde{q}(s)\| + \|\Sigma(s,q(s)) - \tilde{\Sigma}(s,\tilde{q}(s))\|\Big)\,ds \\
&\leqslant Me^{\rho(\tau-t)}\|\eta-\tilde{\eta}\| \\
&+ \ell M\int_t^\tau e^{\rho(s-t)}\Big(\|\Sigma(s,q(s))-\tilde{\Sigma}(s,q(s))\|+(1+\kappa)\|q(s)-\tilde{q}(s)\|\Big)\,ds \\
&\leqslant Me^{\rho(\tau-t)}\|\eta-\tilde{\eta}\| \\
&+ \ell M\int_t^\tau e^{\rho(s-t)}\Big((1+\kappa)\|q(s)-\tilde{q}(s)\| + \|\Sigma-\tilde{\Sigma}\|\|q(s)\|\Big)\,ds.
\end{aligned}$$

Then, due to (**??**),

$$\begin{aligned}\|q(t)-\tilde{q}(t)\| \leqslant & Me^{\rho(\tau-t)}\|\eta-\tilde{\eta}\|+\ell M(1+\kappa)\int_t^\tau e^{\rho(s-t)}\|q(s)-\tilde{q}(s)\|ds \\ & + \ell M^2\|\eta\|\|\Sigma-\tilde{\Sigma}\|\int_t^\tau e^{(\rho+M\ell(1+\kappa))(\tau-s)}e^{\rho(s-t)}ds \\ \leqslant & Me^{\rho(\tau-t)}\|\eta-\tilde{\eta}\|+\ell M(1+\kappa)\int_t^\tau e^{\rho(s-t)}\|q(s)-\tilde{q}(s)\|ds \\ & + \frac{M\|\eta\|}{(1+\kappa)}\|\Sigma-\tilde{\Sigma}\|e^{(\rho+M\ell(1+\kappa))(\tau-t)},\end{aligned} \tag{16}$$

and, by Grönwall's Lemma, for $t \leqslant \tau$,

$$\|q(t)-\tilde{q}(t)\| \leqslant M\left[\|\eta-\tilde{\eta}\|+\frac{\|\eta\|}{1+\kappa}\|\Sigma-\tilde{\Sigma}\|\right]e^{(\rho+2M\ell(1+\kappa))(\tau-t)},. \tag{17}$$

Finally, we now discuss bounds of the function G. Indeed, equations (**??**) and (**??**) imply

$$
\begin{aligned}
&\|G(\Sigma)(\tau,\eta) - G(\tilde{\Sigma})(\tau,\tilde{\eta})\| \\
&\leqslant M\int_{-\infty}^{\tau} e^{-\gamma(\tau-s)}\|f(s,q(s)+\Sigma(s,q(s)))-f(s,\tilde{q}(s)+\tilde{\Sigma}(s,\tilde{q}(s)))\|_X\,ds \\
&\leqslant \ell M \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)} \Big((1+\kappa)\|q(s)-\tilde{q}(s)\| + \|\Sigma-\tilde{\Sigma}\|\|q(s)\|\Big)\,ds \\
&\leqslant \ell M^2(1+\kappa)\left[\|\eta-\tilde{\eta}\|+\frac{\|\eta\|}{1+\kappa}\|\Sigma-\tilde{\Sigma}\|\right]\int_{-\infty}^{\tau} e^{-(\gamma-\rho-2M\ell(1+\kappa))(\tau-s)}\,ds \\
&\quad + \ell M^2\|\eta\|\|\Sigma-\tilde{\Sigma}\|\int_{-\infty}^{\tau} e^{-(\gamma-\rho-M\ell(1+\kappa))(\tau-s)}\,ds
\end{aligned}
$$

Due to (5) and upcoming choice of κ, we obtain that $\gamma-\rho-2\ell M(1+\kappa)>0$ and the above integrals are convergent.

Thus,

$$\begin{aligned}\|G(\Sigma)(\tau,\eta)-G(\tilde{\Sigma})(\tau,\tilde{\eta})\| &\leqslant \frac{\ell M^2\|\eta\|}{\gamma-\rho-\ell M(1+\kappa)}\|\Sigma-\tilde{\Sigma}\| \\ &\quad+\frac{\ell M^2(1+\kappa)}{\gamma-\rho-2\ell M(1+\kappa)}\left[\|\eta-\tilde{\eta}\|+\frac{\|\eta\|}{1+\kappa}\|\Sigma-\tilde{\Sigma}\|\right] \\ &\leqslant \frac{\ell M^2(1+\kappa)}{\gamma-\rho-2\ell M(1+\kappa)}\|\eta-\tilde{\eta}\|+\frac{2\ell M^2}{\gamma-\rho-2\ell M(1+\kappa)}\|\Sigma-\tilde{\Sigma}\|\|\eta\|,\end{aligned}$$

where the denominators are positive, due to (5). Consequently,

$$\|G(\Sigma)(\tau,\eta)-G(\tilde{\Sigma})(\tau,\tilde{\eta})\| \leqslant \kappa\|\eta-\tilde{\eta}\|+\nu\|\Sigma-\tilde{\Sigma}\|\|\eta\|, \qquad (18)$$

in case that

$$\frac{\ell M^2(1+\kappa)}{\gamma-\rho-2\ell M(1+\kappa)} \leqslant \kappa, \qquad (19a)$$

$$\frac{2\ell M^2}{\gamma-\rho-2\ell M(1+\kappa)} < 1. \qquad (19b)$$

Now, (**??**) can be rewritten as $2M\kappa^2+(M^2+M-(\gamma-\rho)/\ell)\kappa+M^2\leqslant 0$, which can be seen as a quadratic polynomial (in κ), admitting two real roots (due to (5)) given by

$$\kappa_\pm := \frac{\frac{\gamma-\rho}{\ell} - M^2 - 2M \pm \sqrt{(\frac{\gamma-\rho}{\ell} - M^2 - 2M)^2 - 8M^3}}{4M}. \qquad (20)$$

Moreover, the condition $(\gamma-\rho)/\ell > M^2 + 2M + \sqrt{8M^3}$ in (5) implies that $(\gamma-\rho)/\ell > M^2 + 2M$ and thus $\kappa_+ > \kappa_- > 0$. Thus, (**??**) is satisfied for any $\kappa \in [\kappa_-, \kappa_+]$.

Equation (**??**) holds true for κ_-, due to $(\gamma-\rho)/\ell > 3M^2 + 2M$ in (5). Moreover, we can isolate κ in (**??**), and thus this inequality is satisfied for any $\kappa < \kappa_* := (\gamma-\rho)/(2M\ell) - M - 1$.

Due to (5), $(\gamma-\rho)/\ell > 3M^2 + 2M$ and $\kappa_- < \kappa_*$. Therefore, both conditions (**??**) are satisfied for any $\kappa \in [\kappa_-, \min\{\kappa_+, \kappa_*\})$.

Consequently, inequality (**??**) with $\Sigma = \tilde{\Sigma}$ implies that the image of the map G lies in $\mathcal{LB}_\Sigma(D)$ and, inequality (**??**) with $\eta = \tilde{\eta}$, shows that G is a contraction.

Therefore, the map G has a unique fixed point, $G(\Sigma^*) = \Sigma^*$. This establishes the existence of the invariant manifold and its invariance.

Furthermore, Σ^* being Lipschitz with constant $\kappa > 0$ and $\Sigma(t, 0) = 0$, together with (**??**), implies the growth estimate (8) within the invariant manifold.

This completes the first part of the proof.

We now embark in the second part of the proof.

For $(t,u)\in\mathbb{R}\times X$, define $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t,u)$.

We show that $\mathcal{M}(t) = \{Im(P_{\Sigma^*}(t)) : t\in\mathbb{R}\}$ is has the property that any solution satisfies (9) (exponential attraction if $\delta > 0$), and thus we wish to bound $\xi(t) := T(t,\tau)u - P_{\Sigma^*}(t)T(t,\tau)u$ for any $\eta\in X$ and $t \geqslant \tau$.

Note that $\xi(t) = p(t) - \Sigma^*(t,q(t))$ due to the definitions in (**??**).

Define $q^*(s,t)$, for $s\leqslant t$, as

$$q^*(s,t) := L(s,t)q(t) + \int_t^s L(s,r)Q(r)f(r,q^*(r,t)+\Sigma^*(r,q^*(r,t)))dr. \quad (21)$$

Since f, Σ^* are Lipschitz with constants $\ell, \kappa > 0$, for $s \leqslant t$,

$$\begin{aligned} &\|q^*(s,t) - q(s)\| \\ &\leqslant M\int_s^t e^{\rho(r-s)}\|f(r,q^*(r,t)+\Sigma^*(r,q^*(r,t)))-f(r,q(r)+p(r))\|dr \\ &\leqslant M\ell \int_s^t e^{\rho(r-s)}(\|\Sigma^*(r,q^*(r,t)) - p(r)\| + \|q^*(r,t) - q(r)\|)dr, \\ &\leqslant M\ell\int_s^t e^{\rho(r-s)}(\|\Sigma^*(r,q(r))-p(r)\|+(1+\kappa)\|q^*(r,t)-q(r)\|)dr. \end{aligned} \tag{22}$$

Hence, by Gronwall's Lemma and definition of ξ,

$$\|q^*(s,t)-q(s)\| \leqslant M\ell \int_s^t e^{(\rho+M\ell(1+\kappa))(r-s)}\|\xi(r)\|dr, \quad s \leqslant t. \quad (23)$$

Also, for $s \leqslant \tau \leqslant t$, we obtain

$$\begin{aligned}&\|q^*(s,t)-q^*(s,\tau)\| \leqslant \|L(s,\tau)Q(\tau)[q^*(\tau,t)-q(\tau)]\|\\
&+\|\int_\tau^s L(s,r)Q(r)[f(r,q^*(r,t)+\Sigma^*(r,q^*(r,t)))-f(r,q^*(r,\tau)+\Sigma^*(r,q^*(r,\tau))]\|dr\\
&\leqslant M^2\ell e^{\rho(\tau-s)}\int_\tau^t e^{(\rho+M\ell(1+\kappa))(r-\tau)}\|\xi(r)\|dr\\
&+M\ell(1+\kappa)\int_s^\tau e^{\rho(r-s)}\|q^*(r,t)-q^*(r,\tau)\|dr,\end{aligned}$$

and by Grönwall's Lemma

$$\|q^*(s,t)-q^*(s,\tau)\| \leqslant M^2\ell \int_\tau^t e^{(\rho+M\ell(1+\kappa))(r-s)}\|\xi(r)\|dr. \quad (24)$$

Now, we use these inequalities to estimate $\|\xi(t)\|$. Note that

$$\xi(t)-L(t,\tau)(I-Q(\tau))\xi(\tau)=p(t)-L(t,\tau)p(\tau)-\Sigma^*(t,q(t))+L(t,\tau)\Sigma^*(\tau,q(\tau))$$

$$=\int_\tau^t L(t,s)(I-Q(s))f(s,q(s)+p(s))ds-\int_{-\infty}^t L(t,s)(I-Q(s))f(s,q^*(s,t))+\Sigma^*(s,q^*(s,t)))ds$$

$$+\int_{-\infty}^\tau L(t,s)(I-Q(s))f(s,q^*(s,\tau)+\Sigma^*(s,q^*(s,\tau)))ds$$

$$=\int_\tau^t L(t,s)(I-Q(s))[f(s,q(s)+p(s))-f(s,q^*(s,t))+\Sigma^*(s,q^*(s,t)))]ds$$

$$-\int_{-\infty}^\tau L(t,s)(I-Q(s))[f(s,q^*(s,t)+\Sigma^*(s,q^*(s,t)))-f(s,q^*(s,\tau)+\Sigma^*(s,q^*(s,\tau)))]ds.$$

Thus, using (**??**) and (**??**), we obtain

$$
\begin{aligned}
&\|\xi(t)-L(t,\tau)(I-Q(\tau))\xi(\tau)\|\\
&\leqslant M\ell\int_\tau^t e^{-\gamma(t-s)}\left(\|p(s)-\Sigma^*(s,q^*(s,t))\|+\|q(s)-q^*(s,t)\|\right)ds\\
&+M\ell(1+\kappa)\int_{-\infty}^t e^{-\gamma(t-s)}\|q^*(s,\tau)-q^*(s,t)\|ds\\
&\leqslant M\ell\int_\tau^t e^{-\gamma(t-s)}\|\xi(s)\|ds\\
&+M^2\ell^2(1+\kappa)\int_\tau^t e^{-\gamma(t-r)}\|\xi(r)\|\int_\tau^r e^{-(\gamma-\rho-M\ell(1+\kappa))(r-s)}dsdr\\
&+M^3\ell^2(1+\kappa)\int_\tau^t e^{-\gamma(t-r)}e^{-(\gamma-\rho-M\ell(1+\kappa))(r-\tau)}\|\xi(r)\|\int_{-\infty}^\tau e^{-\gamma(\tau-s)}e^{(\rho+M\ell(1+\kappa))(\tau-s)}dsdr\\
&\leqslant M\ell\int_\tau^t e^{-\gamma(t-s)}\|\xi(s)\|ds+\frac{M^2\ell^2(1+\kappa)}{\gamma-\rho-M\ell(1+\kappa)}\int_\tau^t e^{-\gamma(t-r)}\|\xi(r)\|dr\\
&+\frac{M^3\ell^2(1+\kappa)}{\gamma-\rho-M\ell(1+\kappa)}\int_\tau^t\|\xi(r)\|e^{-\gamma(t-r)}dr
\end{aligned}
$$

and we have that

$$\|\xi(t)-L(t,\tau)(I-Q(\tau))\xi(\tau)\|\leqslant\left[M\ell+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\right]\int_\tau^t e^{-\gamma(t-r)}\|\xi(r)\|dr.$$

Thus,

$$\|\xi(t)\|\leqslant Me^{-\gamma(t-\tau)}\|\xi(\tau)\|+\left[M\ell+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\right]\int_\tau^t e^{-\gamma(t-r)}\|\xi(r)\|dr$$

and

$$e^{\gamma t}\|\xi(t)\|\leqslant Me^{\gamma\tau}\|\xi(\tau)\|+\left[M\ell+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\right]\int_\tau^t e^{\gamma r}\|\xi(r)\|dr.$$

By Grönwall's Lemma, we obtain the bound in equation (9). $\square$

Teorema

Suppose that the linear evolution process $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential splitting, with constant* $M \geqslant 1$, *exponents* $\gamma > \rho$ *and a family of projections* $\{Q(t) : \ t \in \mathbb{R}\}$.
If $(\gamma - \rho)/\ell$ *satisfies* (5), *then there is a continuous function*

$$\begin{aligned} \Theta^* : \mathbb{R} \times X &\to X \\ (t,u) &\mapsto \Theta^*(t,u), \end{aligned} \tag{25}$$

such that $\Theta^*(t,u) = \Theta^*(t,(I - Q(t))u) = Q(t)\Theta^*(t,u)$, *and* $\Theta^*(t,0) = 0$ *for all* $t \in \mathbb{R}$, *which is uniformly Lipschitz with constant* $\kappa = \kappa(\gamma, \rho, \ell, M) > 0$, *i.e.,*
$\|\Theta^*(t,u) - \Theta^*(t,\tilde{u})\| \leqslant \kappa \|u - \tilde{u}\|$ *for all* $(t,u), (t,\tilde{u}) \in \mathbb{R} \times X$.

Moreover, if $P_{\Theta^*}(t)u := \Theta^*(t,(I-Q(t))u) + (I-Q(t))u$, *for all* $(t,u) \in \mathbb{R} \times X$, *the family given by*

$$\{Im(P_{\Theta^*}(t)) : t \in \mathbb{R}\} := \{\{P^*_{\Theta}(t,u) : u \in X\} : t \in \mathbb{R}\}, \tag{26}$$

is positively invariant such that

$$\|T(t,\tau)P_{\Theta^*}(\tau)u\| \leqslant M(1+\kappa)e^{-(\gamma - M\ell(1+\kappa))(t-\tau)}\|P_{\Theta^*}(\tau)u\|, \tag{27}$$

$t \geqslant \tau$, $u \in X$, *and*

$$\|u - P_{\Theta^*}(\tau)u\| \leqslant Me^{\hat{\delta}(t-\tau)}\|(I - P_{\Theta^*}(t))T(t,\tau)u\|, \tag{28}$$

$t \geqslant \tau, u \in X$, *where* $\hat{\delta} = \rho + M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}$.
Furthermore, if $\gamma - M\ell(1+\kappa) > 0$, $\{Im(P_{\Theta^*}(t)) : t \in \mathbb{R}\}$ *is the stable manifold of the inertial manifold* $\{Im(P_{\Sigma^*}(t)) : t \in \mathbb{R}\}$.

Proof: Given $\kappa > 0$ consider the complete metric space

$$\begin{aligned}\mathcal{LB}_\Theta(\kappa) = \Big\{&\Theta \in C(\mathbb{R}\times X, X) : \|\Theta(t,u) - \Theta(t,\tilde{u})\| \leqslant \kappa \|u - \tilde{u}\|, \Theta(t,0) = 0, \\ &\Theta(t,u) = \Theta(t,(I - Q(t))u) \in Im(Q(t)),\ \forall (t,u),(t,\tilde{u}) \in \mathbb{R}\times X\Big\}.\end{aligned} \tag{29}$$

with the metric $\|\Theta - \tilde{\Theta}\| = \sup\limits_{t\in\mathbb{R}} \sup\limits_{u\neq 0} \dfrac{\|\Theta(t,u) - \tilde{\Theta}(t,u)\|}{\|u\|}$.

Next we outline the heuristic procedure that will establish the way of proving that the invariant manifold is given as a graph of a map in $\mathcal{LB}_\Theta(\kappa)$. We are looking for $\Theta \in \mathcal{LB}_\Theta(\kappa)$ with the property that, if $(\tau, \eta) \in \mathbb{R} \times X$, then a solution u of (2), with initial data $u(\tau) = \Theta(\tau, (I - Q(\tau))\eta) + (I - Q(\tau))\eta \in X$, can be decomposed as $u(t) = q(t) + p(t)$, where $q(t) = \Theta(t, p(t))$ for all $t \geqslant \tau$. Thus, q and p must satisfy, for $t \geqslant \tau$,

$$q(t) = L(t,\tau)Q(\tau)\eta + \int_\tau^t L(t,s)Q(s)f(s, p(s) + \Theta(s, p(s)))ds, \tag{30a}$$

$$p(t) = L(t,\tau)(I - Q(\tau))\eta + \int_\tau^t L(t,s)(I - Q(s))f(s, p(s) + \Theta(s, p(s)))ds. \tag{30b}$$

It follows that

$$\begin{aligned}\|p(t)\| &\leqslant \|L(t,\tau)(I - Q(\tau))\eta\| \\ &\quad + \int_\tau^t \|L(t,s)(I - Q(s))f(s, p(s) + \Theta(s, p(s)))\| ds \\ &\leqslant Me^{-\gamma(t-\tau)}\|\eta\| + \int_\tau^t M\ell e^{-\gamma(t-s)}(1+\kappa)\|p(s)\| ds.\end{aligned}$$

Using Grownwall's inequality,

$$\|p(t)\| \leqslant Me^{-(\gamma - M\ell(1+\kappa))(t-\tau)}\|\eta\|.$$

From this and from the fact that $q(t) = \Theta(p(t))$, we conclude that

$$\begin{aligned}\|L(\tau,t)Q(t)q(t)\| &= \|L(\tau,t)\Theta(p(t))\| \\ &\leqslant \kappa M^2 e^{-(\gamma-\rho-M\ell(1+\kappa))(t-\tau)}\|\eta\|.\end{aligned}$$

Applying $L(\tau, t)Q(t)$ to (15), using that $\Theta(\tau, (I - Q(\tau))\eta) = Q(\tau)\eta$ and making $t \to \infty$ we have

$$0 = \Theta(\tau, (I - Q(\tau))\eta) + \int_{\tau}^{\infty} L(\tau, s)Q(s)f(s, p(s) + \Theta(s, p(s)))ds.$$

Inspired by this we define the operator $\tilde{G} : \mathcal{LB}_{\Theta}(\kappa) \to \mathcal{LB}_{\Theta}(\kappa)$ by

$$\tilde{G}(\Theta)(\tau, \eta) = -\int_{\tau}^{\infty} L(\tau, s)Q(s)f(s, p(s) + \Theta(s, y(s)))ds, \quad (31)$$

$(\tau, \eta) \in \mathbb{R} \times X$.

The fact that $\tilde{G}$ is a well-defined contraction is similar to Theorem 1, and we refrain from giving a proof. Hence $\tilde{G}$ admits a unique fixed point $\Theta^* \in \mathcal{LB}_\Theta(\kappa)$ satisfying the desired properties.
We now embark in the proof of (13). For any $(\tau, \eta) \in X$ and $t \geqslant \tau$.

$$p(t) = L(t,\tau)(I - Q(\tau))\eta + \int_\tau^t L(t,s)(I - Q(s))f(s, q(s) + p(s))ds$$

and thus we wish to bound the variable
$\eta(t) := T(t,\tau)u - P_{\Theta^*}(t)T(t,\tau)u$ for any $u \in X$ and $t \geqslant \tau$. Note that $\eta(t) = q(t) - \Theta^*(t, p(t))$ due to the definitions in (15).

Define $p^*(s,t)$, for $s \geqslant t$, as

$$\begin{aligned}p^*(s,t) := & L(s,t)p(t) \\ & + \int_t^s L(s,r)(I - Q(r))f(r, \Theta^*(r, p^*(r,t)) + p^*(r,t))dr.\end{aligned} \tag{32}$$

Since f, Θ^* are Lipschitz with respective constants $\ell, \kappa > 0$, we obtain

$$\begin{aligned}&\|p^*(s,t) - p(s)\| \\ &\leqslant M\int_t^s e^{-\gamma(s-r)}\|f(r,\Theta^*(r,p^*(r,t))+p^*(r,t)) - f(r,q(r)+p(r))\|dr \\ &\leqslant M\ell\int_t^s e^{-\gamma(s-r)}(\|q(r) - \Theta^*(r,p(r))\| + (1+\kappa)\|p^*(r,t) - p(r)\|)dr,\end{aligned} \tag{33}$$

and, by Grönwall's Lemma,

$$\|p^*(s,t) - p(s)\| \leqslant M\ell \int_t^s e^{-(\gamma - M\ell(1+\kappa))(s-r)}\|\eta(r)\|dr. \tag{34}$$

Also, for $s \geqslant t \geqslant \tau$, we obtain

$$\|p^*(s,\tau) - p^*(s,t)\| \leqslant \|L(s,t)(I - Q(t))[p^*(t,\tau) - p(t)]\|$$
$$+\|\int_t^s L(s,r)(I - Q(r))[f(r,\Theta^*(r,p^*v(r,\tau)) + p^*(r,\tau)) - f(r,\Theta^*(r,p^*(r,t)) + p^*(r,t))]\|dr$$
$$\leqslant M^2\ell e^{-\gamma(s-t)} \int_\tau^t e^{-(\gamma - M\ell(1+\kappa))(t-r)}\|\eta(r)\|dr$$
$$+ M\ell(1+\kappa)\int_t^s e^{-\gamma(s-r)}\|p^*(r,\tau) - p^*(r,t)\|dr,$$

and again by Grönwall's Lemma,

$$\|p^*(s,\tau) - p^*(s,t)\| \leqslant M^2\ell \int_\tau^t e^{-(\gamma - M\ell(1+\kappa))(s-r)}\|\eta(r)\|dr. \quad (35)$$

Now, we use these inequalities to estimate $\|\eta(\tau)\|$. Note that

$$\eta(\tau)-L(\tau,t)Q(t)\eta(t)=q(\tau)-L(\tau,t)q(t)-\Theta^*(\tau,p(\tau))+L(\tau,t)\Theta^*(t,p(t))$$

$$=\int_t^\tau L(\tau,s)Q(s)[f(s,q(s)+p(s))-f(s,\Theta^*(s,p^*(s,\tau))+p^*(s,\tau))]ds$$

$$+\int_t^\infty L(\tau,s)Q(s)[f(s,\Theta^*(s,p^*(s,\tau))+p^*(s,\tau))-f(s,\Theta^*(s,p^*(s,t))+p^*(s,t))]ds.$$

Thus, using (19) and (20), we obtain

$$
\begin{aligned}
&\|\eta(\tau) - L(\tau,t)Q(t)\eta(t)\| \\
&\leqslant M\ell \int_\tau^t e^{-\rho(\tau-s)} \left(\|q(s) - \Theta^*(s,p^*(s,\tau))\| + \|p(s) - p^*(s,\tau)\|\right) ds \\
&+ M\ell(1+\kappa) \int_t^\infty e^{-\rho(\tau-s)} \|p^*(s,t) - p^*(s,\tau)\| ds \\
&\leqslant M\ell \int_\tau^t e^{-\rho(\tau-s)} \|\eta(s)\| ds \\
&+ M^2\ell^2(1+\kappa) \int_\tau^t e^{-(\gamma-\rho-M\ell(1+\kappa))(s-\tau)} \int_\tau^s e^{-(\gamma-M\ell(1+\kappa))(\tau-r)} \|\eta(r)\| drds \\
&+ M^3\ell^2(1+\kappa) \int_t^\infty e^{-(\gamma-\rho-M\ell(1+\kappa))(s-\tau)} \int_\tau^t e^{-(\gamma-M\ell(1+\kappa))(\tau-r)} \|\eta(r)\| drds.
\end{aligned}
$$

Hence

$$\|\eta(\tau) - L(\tau,t)Q(t)\eta(t)\| \leqslant M\ell\int_\tau^t e^{-\rho(\tau-s)}\|\eta(s)\|ds$$
$$+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\int_\tau^t e^{-(\gamma-\rho-M\ell(1+\kappa))(\tau-r)}e^{-\rho(\tau-r)}\|\eta(r)\|\,dr$$

and we have that

$$\|\eta(\tau)-L(\tau,t)Q(t)\eta(t)\| \leqslant \left[M\ell+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\right]\int_\tau^t e^{-\rho(\tau-r)}\|\eta(r)\|dr.$$

Thus,

$$\|\eta(\tau)\| \leqslant Me^{-\rho(\tau-t)}\|\eta(t)\|+\left[M\ell+\frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}\right]\int_\tau^t e^{-\rho(\tau-r)}\|\eta(r)\|dr.$$

By Grönwall's Lemma, we obtain the bound in (13). $\square$

The saddle point property

We now obtain the saddle point property as an immediate consequence of Theorems 1 and 2. We define the **unstable** and **stable sets** of a hyperbolic global solution u_* of (2) as

$$W^u(u_*) := \left\{ (\tau, u_0) \in \mathbb{R} \times X : \begin{array}{l} \text{there is a solution } u : (-\infty, \tau] \to X \\ \text{such that } u(\tau) = u_0 \text{ and} \\ \lim_{t \to -\infty} \|u(t) - u_*(t)\|_X = 0 \end{array} \right\} \tag{36a}$$

$$W^s(u_*) := \left\{ (\tau, u_0) \in \mathbb{R} \times X : \begin{array}{l} \text{there is a solution } u : [\tau, \infty) \to X \\ \text{such that } u(\tau) = u_0 \text{ and} \\ \lim_{t \to +\infty} \|u(t) - u_*(t)\|_X = 0 \end{array} \right\} \tag{36b}$$

Corolário

Suppose that the linear evolution process $\{L(t,\tau) : t \geqslant \tau\}$ has exponential dichotomy, with constant $M \geqslant 1$, exponent $\gamma > 0$ and a family of projections $\{Q(t) : \ t \in \mathbb{R}\}$.
Suppose that $\ell > 0$ is sufficiently small, then there are continuous functions $\Sigma^u \in \mathcal{L}_\Sigma(\kappa)$ and $\Theta^s \in \mathcal{L}_\Theta(\kappa)$ such that the unstable and stable manifolds of $u_* = 0$ are given by

$$W^u(0) = \{(\tau, u) \in \mathbb{R} \times X : u = Q(\tau)u + \Sigma^u(\tau, Q(\tau)u)\}, \tag{37a}$$
$$W^s(0) = \{(\tau, u) \in \mathbb{R} \times X : u = \Theta^s(\tau, (I - Q(\tau))u) + (I - Q(\tau))u\}. \tag{37b}$$

Moreover, solutions within the unstable (resp. stable) manifold exponentially decay to zero backwards (resp. forwards) in time, according to (8) and (12).

Proof: For $\ell > 0$ sufficiently small, the condition (5) is satisfied and $\delta > 0$, and thus we obtain the graph of Σ^* from Theorem 1. We now prove that the unstable set $W^u(0)$ defined in (21a) coincides with the graph of $\Sigma^u := \Sigma^*$. On one hand, the graph of Σ^u is contained in the unstable set by (8). On the other hand, any solution $z : (-\infty, t] \to X$ which backwards converges to zero satisfies, from (9),

$$\begin{aligned}\|z(t) - P_{\Sigma^*}(t)z(t)\| &= \|(I - Q(t))z(t) - \Sigma^u(t, Q(t)z(t))\| \\ &\leqslant M\|(I - P_{\Sigma^*}(\tau))z(\tau)\|e^{-\delta(t-\tau)}, \quad t \geqslant \tau.\end{aligned}$$

Since $\delta > 0$, we obtain that $(I - Q(t))z(t) = \Sigma^u(t, Q(t)z(t))$ for all $t \in \mathbb{R}$ as $\tau \to -\infty$, and thus any element in the unstable set lies in the graph of Σ^u. The case of stable manifold is analogous applying Theorem 2. $\square$

Roughness of Exponential Dichotomy

We now prove that the roughness of exponential dichotomy, i.e., that exponential dichotomies are preserved under perturbations. Assume that the linear evolution process $\{L(t,\tau) : t \geqslant \tau\}$ associated to the problem

$$\dot{u} = A(t)u, \;\; t \geqslant \tau, \quad u(\tau) = u_0. \tag{38}$$

has exponential dichotomy with constant M and exponent $\gamma > 0$ and consider the linear evolution process $\{T(t,\tau) : t \geqslant \tau\}$, associated to a perturbation of it, given by the linear equation,

$$\dot{u} = A(t)u + B(t)u, \;\; t \geqslant \tau, \quad u(\tau) = u_0. \tag{39}$$

where the map $t \mapsto B(t) \in \mathcal{L}(X)$ is strongly continuous for $t \in \mathbb{R}$ and $\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant \ell$, for some suitably small $\ell > 0$.

Recall that, as in (4), the evolution process $\{T(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ associated to (24) is given by

$$T(t,\tau) = L(t,\tau) + \int_\tau^t L(t,s)B(s)T(s,\tau)\,ds, \quad t \geqslant \tau. \tag{40}$$

We wish to prove that (24) has exponential dichotomy for suitably small ℓ.

This result can be obtained by firstly applying Theorems 1 and 2 in a linear setting, which are suitable in order to establish the existence of the linear invariant manifold and its stable manifold (see Corollary 2) and then apply it to (24) with $\gamma > 0$ and $\rho = -\gamma$.

Corolário

If $\{L(t,\tau) : t \geqslant \tau\}$ has exponential splitting with constant M, exponents $\gamma > \rho$ and projections $\{Q(t) : \ t \in \mathbb{R}\}$ and (5) is satisfied, then

- *There are maps $\Sigma^*, \Theta^* : \mathbb{R} \times X \to X$, $\Sigma^*(t,\cdot), \Theta^*(t,\cdot) \in \mathcal{L}(X)$ and $\|\Sigma^*(t,u)\| \leqslant \kappa \|u\|_X$, $\|\Theta^*(t,u)\|_X \leqslant \kappa \|u\|_X$ for all $(t,u) \in \mathbb{R} \times X$ and for some $\kappa = \kappa_\ell > 0$;*
- *The graph $\mathcal{G}(\Sigma^*)$ of Σ^* is an invariant family and (9) holds, the graph $\mathcal{G}(\Theta^*)$ of Θ^* is a positively invariant family;*
- *The evolution process $\{T(t,\tau) : t \geqslant \tau\}$ given by (25) satisfies*
$$\begin{aligned} \|T(t,\tau)P_{\Sigma^*}(\tau)\| &\leqslant M(1+\kappa)e^{-(\rho+M\ell(1+\kappa))(t-\tau)}, \ t \leqslant \tau, \\ \|T(t,\tau)P_{\Theta^*}(\tau)\| &\leqslant M(1+\kappa)e^{-(\gamma-M\ell(1+\kappa))(t-\tau)}, \ t \geqslant \tau, \end{aligned} \tag{41}$$
where $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u)$ and $P_{\Theta^*}(t)u := \Theta^*(t,(I-Q(t))u) + (I-Q(t))u$, $t \in \mathbb{R}$.

Proof: The proof is a direct consequence of Theorems 1 and 2 in the case that $f(t,\cdot)$ is linear and uniformly (with respect to t) bounded. Note that, the linearity of $\Sigma(t,\cdot)$ follows since $f(t,\cdot)$ is linear, and thereby $G(\Sigma)$ given by equation (**??**) is also linear. Consequently, the fixed point, $G(\Sigma^*)(t,u) = \Sigma^*(t,u)$, is linear. Similarly, $\tilde{G}$ in equation (16) is also linear and so is Θ^*. □

Next, we show the robustness of the exponential dichotomy.

Corolário

Suppose that $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential dichotomy with constant* $M \geqslant 1$, *exponent* $\gamma > 0$ *and family of projections* $\{Q(t) : t \in \mathbb{R}\}$. *If* $\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant \ell$, *where* $0<\ell<\frac{2\gamma}{3M(M+1)}$, *then* $\{T(t,\tau) : t \geqslant \tau\}$ *has exponential dichotomy, that is, there are projections* $\{Q_\ell(t) : t \in \mathbb{R}\}$ *with* $T(t,\tau) : Im(Q_\ell(\tau)) \to Im(Q_\ell(t))$ *being an isomorphism,* $t \geqslant \tau$, *and*

$$\begin{aligned} \|T(t,\tau)Q_\ell(\tau)\|_{\mathcal{L}(X)} &\leqslant M_\ell e^{\gamma_\ell(t-\tau)}, \quad t \leqslant \tau \\ \|T(t,\tau)(I-Q_\ell(\tau))\|_{\mathcal{L}(X)} &\leqslant M_\ell e^{-\gamma_\ell(t-\tau)}, \quad t \geqslant \tau, \end{aligned} \tag{42}$$

where $M_\ell := M(1+\kappa_\ell)/(1-2\kappa_\ell) > 1$ *and* $\gamma_\ell := \gamma - \ell M(1+\kappa_\ell) > 0$ *for the Lipschitz constant* κ_ℓ *obtained in Corollary 2.*

Moreover,

$$\sup_{t\in\mathbb{R}} \|Q(t)-Q_\ell(t)\|_{\mathcal{L}(X)} \leqslant \frac{2\kappa_\ell}{1-2\kappa_\ell}. \tag{43}$$

Proof: If $P_{\Theta^*}(t)u := (I - Q(t))u + \Theta^*(t, (I - Q(t))u)$ and $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u)$, for $(t, u) \in \mathbb{R} \times X$, where Σ^* and Θ^* are the bounded linear maps obtained in Corollary 2, with norm less than $\kappa_\ell > 0$. We will prove that $X = Im(P_{\Sigma^*}(t)) \oplus Im(P_{\Theta^*}(t))$, for every $t \in \mathbb{R}$. That is, we show that, for each $(t, u) \in \mathbb{R} \times X$,

$$\begin{aligned} &\mathcal{I}_u(t) : X \to X \\ &\qquad\qquad v \mapsto \mathcal{I}_u(t)v := u - \Sigma^*(t, v) - \Theta^*(t, v), \end{aligned} \tag{44}$$

has a unique fixed point. If that is the case, for each $(t, u) \in \mathbb{R} \times X$, there exists a unique $v_u \in X$ such that $\mathcal{I}_u(t)v_u = v_u$, that is,

$$u - \Sigma^*(t, v_u) - \Theta^*(t, v_u) = v_u = Q(t)v_u + (I - Q(t))v_u, \text{ or} \tag{45}$$

$$\begin{aligned} u &= Q(t)v_u + \Sigma^*(t, v_u) + (I - Q(t))v_u + \Theta^*(t, v_u) \\ &= P_{\Sigma^*}(t)v_u + P_{\Theta^*}(t)v_u, \end{aligned} \tag{46}$$

Which is the unique representation of u as a sum of elements of $Im(P_{\Sigma^*}(t))$ and $Im(P_{\Theta^*}(t))$ and proves the desired decomposition.

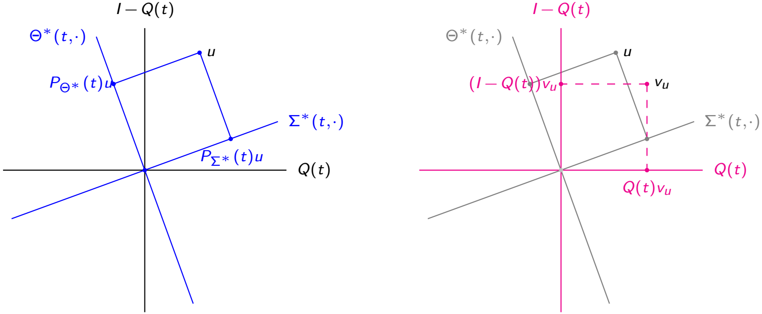

Figure: Given a point $u \in X$, we find a unique point $v_u \in X$ such that $P_{\Sigma^*}(t)u = Q(t)v_u + \Sigma^*(t, Q(t)v_u)$ and $P_{\Theta^*}(t)u = (I - Q(t))v_u + \Theta^*(t, (I - Q(t))v_u)$.

In order to show that $\mathcal{I}_u(t)$ has a unique fixed point, note that $\mathcal{I}_u(t)$ is a contraction on X, since

$$\begin{aligned}\|\mathcal{I}_u(t)v-\mathcal{I}_u(t)\tilde{v}\| &= \|\Sigma^*(t,\tilde{v})-\Sigma^*(t,v)+\Theta^*(t,\tilde{v})-\Theta^*(t,v)\|,\\ &\leqslant 2\kappa\|v-\tilde{v}\|,\end{aligned} \tag{47}$$

for any $v,\tilde{v}\in X$, as the graphs Σ^*,Θ^* are Lipschitz with constant $\kappa=\kappa_\ell>0$. Thus, $\mathcal{I}_u(t)$ is a contraction for each $(t,u)\in\mathbb{R}\times X$, and for all $\kappa\in[\kappa_-,\min\{1/2,\min\{\kappa_+,\kappa_*\}\})$, where κ_- is given by (**??**), since the hypothesis on ℓ in Corollary 3 implies that $\kappa_-<1/2$ and thus any κ as above implies that we have a contraction. Without loss of generality, we may choose $\kappa_\ell:=\kappa_-$. Note that, for each $u\in X$, since v_u is the unique element of X satisfying $v_u=u-\Sigma^*(t,v_u)-\Theta^*(t,v_u)$, the map $u\mapsto v_u$ is a linear bounded operator such that

$$\|v_u\|_X\leqslant\frac{\|u\|_X}{1-2\kappa}. \tag{48}$$

For each $t \in \mathbb{R}$, define $Q_\ell(t) \in \mathcal{L}(X)$ the linear projection onto $R(P_{\Sigma^*}(t))$ along $R(P_{\Theta^*}(t))$, which can be written as $Q_\ell(t)u := P_{\Sigma^*}(t)v_u$ due to the first part of the proof. Its complementary projection is given by $(I - Q_\ell(t))u = P_{\Theta^*}(t)v_u$, for each $(t, u) \in \mathbb{R} \times X$.
From Corollary 2, we have that $\{R(Q_\ell(t)) : t \in \mathbb{R}\}$ is invariant and $\{R(I - Q_\ell(t)) : t \in \mathbb{R}\}$ is positively invariant. Thus $T(t,\tau)Q_\ell(\tau) = Q_\ell(t)T(t,\tau)$, for every $t \geqslant \tau$. Equations (26) and (33) imply the desired bounds (27). This proves that $\{T(t,\tau) : t \geqslant \tau\}$ has exponential dichotomy with constant $M_\ell := M(1+\kappa)/(1-2\kappa)$ and exponent $\gamma_\ell := \gamma - M\ell(1+\kappa) > 0$.

Lastly, we prove the bound in equation (28), that is, the continuous dependence of the projections $\{Q(t) : t \in \mathbb{R}\}$ and $\{Q_\ell(t) : t \in \mathbb{R}\}$, correponding to the exponential dichotomies of the respective evolution processes $\{L(t,\tau) : t \geqslant \tau\}$ and $\{T(t,\tau) : t \geqslant \tau\}$.
Consider $u \in X$, which can be uniquely decomposed as $u = v_u + \Sigma^*(t, v_u) + \Theta^*(t, v_u)$. Hence,
$Q(t)u = Q(t)v_u + \Theta^*(t, v_u)$, since $Q(t)\Sigma^*(t, v_u) = 0$, and
$Q_\ell(t)u = Q(t)v_u + \Sigma^*(t, v_u)$, by definition of $Q_\ell(t)$ and because $Q_\ell(t)\Theta^*(t, v_u) = 0$. Therefore,

$$Q(t)u - Q_\ell(t)u = \Theta^*(t, v_u) - \Sigma^*(t, v_u). \tag{49}$$

Since the maps Σ^*, Θ^* are Lipschitz with constant $\kappa_\ell > 0$, and due to equation (33), we obtain the desired bound (28). $\square$