

Sistemas Dinâmicos Não Lineares

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Exponential Splitting and Dichotomy

Definição (Exponential Splitting)

A linear evolution process $\{L(t, \tau) : t \geq \tau\} \subset \mathcal{L}(X)$ has **exponential splitting**, with constant $M \geq 1$, exponents $\gamma, \rho \in \mathbb{R}$, with $\gamma > \rho$, and a family of projections $\{Q(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$, if

- i) $Q(t)L(t, \tau) = L(t, \tau)Q(\tau)$, for all $t \geq \tau$,
- ii) $L(t, \tau) : \text{Im}(Q(\tau)) \rightarrow \text{Im}(Q(t))$ is an isomorphism, with inverse denoted by $L(\tau, t)$,
- iii) the following estimates hold

$$\begin{aligned}\|L(t, \tau)Q(\tau)\|_{\mathcal{L}(X)} &\leq M e^{-\rho(t-\tau)}, \quad t \leq \tau, \\ \|L(t, \tau)(I - Q(\tau))\|_{\mathcal{L}(X)} &\leq M e^{-\gamma(t-\tau)}, \quad t \geq \tau.\end{aligned}\tag{1}$$

Inertial Manifolds

Consider the following semilinear differential initial value problem

$$\begin{aligned}\dot{u} &= A(t)u + f(t, u), \quad t > \tau, \\ u(\tau) &= u_0 \in X,\end{aligned}\tag{2}$$

with $f : \mathbb{R} \times X \rightarrow X$ continuous, $f(t, 0) = 0$, for all $t \in \mathbb{R}$ and uniformly Lipschitz in the second variable with Lipschitz constant $\ell > 0$, i.e., $\|f(t, u) - f(t, \tilde{u})\| \leq \ell \|u - \tilde{u}\|$ for any $(t, u), (t, \tilde{u}) \in \mathbb{R} \times X$.

Assume that the family of linear operators $\{A(t) : t \in \mathbb{R}\}$ (not bounded) defines a linear evolution process $\{L(t, \tau) : t \geq \tau\} \subset \mathcal{L}(X)$, i.e., for each $(\tau, u_0) \in \mathbb{R} \times X$, the ‘solution’ of the linear problem,

$$\begin{aligned}\dot{u} &= A(t)u, \quad t \geq \tau, \\ u(\tau) &= u_0 \in X,\end{aligned}\tag{3}$$

is given by $u(t, \tau, u_0) = L(t, \tau)u_0$, for $t \geq \tau$, $L(t, t) = Id_X$, $L(t, s)L(s, \tau) = L(t, \tau)$, $t \geq s \geq \tau$ and $[\tau, \infty) \ni t \mapsto L(t, \tau)u_0 \in X$ is continuous, for all $(\tau, u_0) \in \mathbb{R} \times X$.

With this, solutions of (2) define a nonlinear evolution process $\{T(t, \tau) : t \geq \tau\} \subset \mathcal{C}(X)$ given by the variation of constants formula, that is,

$$T(t, \tau)u = L(t, \tau)u + \int_{\tau}^t L(t, s)f(s, T(s, \tau)u) ds, \quad t \geq \tau, u \in X. \tag{4}$$

Teorema

Suppose that the linear evolution process $\{L(t, \tau) : t \geq \tau\}$ has exponential splitting, with constant $M \geq 1$, exponents $\gamma > \rho$ and a family of projections $\{Q(t) : t \in \mathbb{R}\}$. If $f : \mathbb{R} \times X \rightarrow X$ is continuous, $f(t, 0) = 0$, $f(t, \cdot) : X \rightarrow X$ is Lipschitz continuous with Lipschitz constant $\ell > 0$, for all $t \in \mathbb{R}$, and

$$\frac{\gamma - \rho}{\ell} > \max\{M^2 + 2M + \sqrt{8M^3}, 3M^2 + 2M\}, \quad (5)$$

then there is a continuous function

$$\begin{aligned} \Sigma^* : \mathbb{R} \times X &\rightarrow X \\ (t, u) &\mapsto \Sigma^*(t, u) \end{aligned} \quad (6)$$

such that $\Sigma^*(t, u) = \Sigma^*(t, Q(t)u) = (I - Q(t))\Sigma^*(t, u)$ and $\Sigma^*(t, 0) = 0$, for all $t \in \mathbb{R}$.

In addition $\Sigma^*(t, \cdot) : X \rightarrow X$ Lipschitz continuous with Lipschitz constant $\kappa = \kappa(\gamma, \rho, \ell, M) > 0$, for all $t \in \mathbb{R}$, that is,

$$\|\Sigma^*(t, u) - \Sigma^*(t, \tilde{u})\| \leq \kappa \|u - \tilde{u}\|, \text{ for all } (t, u), (t, \tilde{u}) \in \mathbb{R} \times X.$$

Moreover, the graph of $\Sigma^*(t, .)$, for each $t \in \mathbb{R}$, given by

$$\mathcal{M}(t) := \{u \in X : u = q + \Sigma^*(t, q), q \in \text{Im}(Q(t))\}, \quad (7)$$

yields an invariant manifold $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ for the evolution process $\{T(t, \tau) : t \geq \tau\}$ given by (4).

In other words, it is invariant and if

$$P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u), \quad (t, u) \in \mathbb{R} \times X$$

is the nonlinear projection onto $\mathcal{M}(t)$.

(i) $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ has controlled growth: for $(\tau, u) \in \mathbb{R} \times X$, $t \leq \tau$,

$$\|T(t, \tau)P_{\Sigma^*}(\tau)u\| \leq M(1 + \kappa)e^{-(\rho + \ell M(1 + \kappa))(t - \tau)} \|P_{\Sigma^*}(\tau)u\|. \quad (8)$$

(ii) $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ satisfies: for any $(\tau, u) \in \mathbb{R} \times X$ and $t \geq \tau$,

$$\|T(t, \tau)u - P_{\Sigma^*}(t)T(t, \tau)u\| \leq M\|(I - P_{\Sigma^*}(\tau))u\|e^{-\delta(t - \tau)}, \quad (9)$$

where $\delta := \gamma - M\ell - \frac{M^2\ell^2(1 + \kappa)(1 + M)}{\gamma - \rho - \ell M(1 + \kappa)}$. If $\delta > 0$, $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ is an inertial manifold.

Prova: The proof is divided into two parts. First, we show that there is a function Σ^* yielding the graph of the invariant manifold, as desired. Second, we show that this graph is exponentially dominated.

For the first part, given $\kappa > 0$, consider the following complete metric space,

$$\begin{aligned} \mathcal{LB}_\Sigma(\kappa) := & \left\{ \Sigma \in C(\mathbb{R} \times X, X) : \sup_{t \in \mathbb{R}} \frac{\|\Sigma(t, u) - \Sigma(t, \tilde{u})\|}{\|u - \tilde{u}\|} \leq \kappa, \right. \\ & \left. \Sigma(t, 0) = 0, \Sigma(t, u) = \Sigma(t, Q(t)u) \in N(Q(t)), \forall t \in \mathbb{R} \right\} \end{aligned} \quad (10)$$

with the metric $\|\Sigma - \tilde{\Sigma}\| := \sup_{t \in \mathbb{R}} \sup_{u \neq 0} \frac{\|\Sigma(t, u) - \tilde{\Sigma}(t, u)\|}{\|u\|}$.

We are looking for $\Sigma \in \mathcal{LB}_\Sigma(\kappa)$ such that, if $(\tau, \eta) \in \mathbb{R} \times X$, then a solution u of (2) with initial data $u(\tau) = Q(\tau)\eta + \Sigma(\tau, Q(\tau)\eta) \in X$ can be decomposed as $u(t) = q(t) + p(t)$, where $p(t) = \Sigma(t, q(t))$ for all $t \in \mathbb{R}$. Thus, q and p must satisfy

$$q(t) = L(t, \tau)Q(\tau)\eta + \int_{\tau}^t L(t, s)Q(s)f(s, q(s) + \Sigma(s, q(s)))ds, \quad (11a)$$

for $t \leq \tau$ and

$$p(\tau) = L(\tau, t)(I - Q(t))p(t) + \int_t^{\tau} L(\tau, s)(I - Q(s))f(s, q(s) + \Sigma(s, q(s)))ds, \quad (11b)$$

for $t \leq \tau$.

First, let us control the growth of $q(t)$. Since $\{L(t, s) : t \geq s\}$ has exponential splitting, $f(t, 0) = 0$ and f and Σ are Lipschitz with respective constants ℓ and κ , we obtain

$$\|q(t)\| \leq M e^{-\rho(t-\tau)} \|\eta\| + \int_t^\tau \ell M e^{-\rho(t-s)} (1+\kappa) \|q(s)\| ds, \quad t \leq \tau. \quad (12)$$

Then, by Grönwall's Lemma,

$$\|q(t)\| \leq M e^{(\rho+M\ell(1+\kappa))(\tau-t)} \|\eta\|, \quad t \leq \tau. \quad (13)$$

Heuristically, since we wish that $p(t) = \Sigma(t, q(t))$ for $\Sigma \in \mathcal{LB}_\Sigma(\kappa)$, the growth in equation (13) implies that the limit $e^{-\gamma(\tau-t)} \|p(t)\| \rightarrow 0$, as $t \rightarrow -\infty$.

Thus, due to the exponential splitting of $\{L(t, \tau) : t \geq \tau\}$, the first term in (11b) goes to zero as $t \rightarrow -\infty$, yielding

$$p(\tau) = \int_{-\infty}^{\tau} L(\tau, s)(I - Q(s))f(s, q(s)) + \Sigma(s, q(s))ds. \quad (14)$$

Hence, to prove that $\Sigma \in \mathcal{LB}_\Sigma(\kappa)$ that satisfies $p(\tau) = \Sigma(\tau, Q(\tau)\eta)$, it is equivalent to find a fixed point of the following map,

$$G(\Sigma)(\tau, \eta) := \int_{-\infty}^{\tau} L(t, s)(I - Q(s))f(s, q(s)) + \Sigma(s, q(s)))ds. \quad (15)$$

Next, we show that $G : \mathcal{LB}_\Sigma(\kappa) \rightarrow \mathcal{LB}_\Sigma(\kappa)$ is a well defined contraction in the complete metric space $\mathcal{LB}_\Sigma(\kappa)$.

Let $\eta, \tilde{\eta} \in X$, $\Sigma, \tilde{\Sigma} \in \mathcal{LB}_\Sigma(\kappa)$ with corresponding solutions $q(t), \tilde{q}(t)$ of (11a).

Thus, for $t \leq \tau$,

$$\begin{aligned}
 \|q(t) - \tilde{q}(t)\| &\leq M e^{\rho(\tau-t)} \|\eta - \tilde{\eta}\| \\
 &+ M \int_t^\tau e^{\rho(s-t)} \|f(s, q(s) + \Sigma(s, q(s))) - f(s, \tilde{q}(s) + \tilde{\Sigma}(s, \tilde{q}(s)))\| ds \\
 &\leq M e^{\rho(\tau-t)} \|\eta - \tilde{\eta}\| \\
 &+ \ell M \int_t^\tau e^{-\rho(t-s)} \left(\|q(s) - \tilde{q}(s)\| + \|\Sigma(s, q(s)) - \tilde{\Sigma}(s, \tilde{q}(s))\| \right) ds \\
 &\leq M e^{\rho(\tau-t)} \|\eta - \tilde{\eta}\| \\
 &+ \ell M \int_t^\tau e^{\rho(s-t)} \left(\|\Sigma(s, q(s)) - \tilde{\Sigma}(s, \tilde{q}(s))\| + (1+\kappa) \|q(s) - \tilde{q}(s)\| \right) ds \\
 &\leq M e^{\rho(\tau-t)} \|\eta - \tilde{\eta}\| \\
 &+ \ell M \int_t^\tau e^{\rho(s-t)} \left((1+\kappa) \|q(s) - \tilde{q}(s)\| + \|\Sigma - \tilde{\Sigma}\| \|q(s)\| \right) ds.
 \end{aligned}$$

Then, due to (13),

$$\begin{aligned}
 \|q(t) - \tilde{q}(t)\| &\leq M e^{\rho(\tau-t)} \|\eta - \tilde{\eta}\| + \ell M (1+\kappa) \int_t^\tau e^{\rho(s-t)} \|q(s) - \tilde{q}(s)\| ds \\
 &\quad + \ell M^2 \|\eta\| \|\Sigma - \tilde{\Sigma}\| \int_t^\tau e^{(\rho+M\ell(1+\kappa))(\tau-s)} e^{\rho(s-t)} ds \\
 &\leq M e^{\rho(\tau-t)} \|\eta - \tilde{\eta}\| + \ell M (1+\kappa) \int_t^\tau e^{\rho(s-t)} \|q(s) - \tilde{q}(s)\| ds \\
 &\quad + \frac{M \|\eta\|}{(1+\kappa)} \|\Sigma - \tilde{\Sigma}\| e^{(\rho+M\ell(1+\kappa))(\tau-t)},
 \end{aligned} \tag{16}$$

and, by Grönwall's Lemma, for $t \leq \tau$,

$$\|q(t) - \tilde{q}(t)\| \leq M \left[\|\eta - \tilde{\eta}\| + \frac{\|\eta\|}{1+\kappa} \|\Sigma - \tilde{\Sigma}\| \right] e^{(\rho+2M\ell(1+\kappa))(\tau-t)}. \tag{17}$$

Finally, we now discuss bounds of the function G . Indeed, equations (13) and (17) imply

$$\begin{aligned}
 & \|G(\Sigma)(\tau, \eta) - G(\tilde{\Sigma})(\tau, \tilde{\eta})\| \\
 & \leq M \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)} \|f(s, q(s) + \Sigma(s, q(s))) - f(s, \tilde{q}(s) + \tilde{\Sigma}(s, \tilde{q}(s)))\|_X ds \\
 & \leq \ell M \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)} \left((1 + \kappa) \|q(s) - \tilde{q}(s)\| + \|\Sigma - \tilde{\Sigma}\| \|q(s)\| \right) ds \\
 & \leq \ell M^2 (1 + \kappa) \left[\|\eta - \tilde{\eta}\| + \frac{\|\eta\|}{1 + \kappa} \|\Sigma - \tilde{\Sigma}\| \right] \int_{-\infty}^{\tau} e^{-(\gamma - \rho - 2M\ell(1 + \kappa))(\tau-s)} ds \\
 & \quad + \ell M^2 \|\eta\| \|\Sigma - \tilde{\Sigma}\| \int_{-\infty}^{\tau} e^{-(\gamma - \rho - M\ell(1 + \kappa))(\tau-s)} ds
 \end{aligned}$$

Due to (5) and upcoming choice of κ , we obtain that $\gamma - \rho - 2\ell M(1 + \kappa) > 0$ and the above integrals are convergent.

Thus,

$$\begin{aligned} \|G(\Sigma)(\tau, \eta) - G(\tilde{\Sigma})(\tau, \tilde{\eta})\| &\leqslant \frac{\ell M^2 \|\eta\|}{\gamma - \rho - \ell M(1 + \kappa)} \|\Sigma - \tilde{\Sigma}\| \\ &\quad + \frac{\ell M^2(1 + \kappa)}{\gamma - \rho - 2\ell M(1 + \kappa)} \left[\|\eta - \tilde{\eta}\| + \frac{\|\eta\|}{1 + \kappa} \|\Sigma - \tilde{\Sigma}\| \right] \\ &\leqslant \frac{\ell M^2(1 + \kappa)}{\gamma - \rho - 2\ell M(1 + \kappa)} \|\eta - \tilde{\eta}\| + \frac{2\ell M^2}{\gamma - \rho - 2\ell M(1 + \kappa)} \|\Sigma - \tilde{\Sigma}\| \|\eta\|, \end{aligned}$$

where the denominators are positive, due to (5). Consequently,

$$\|G(\Sigma)(\tau, \eta) - G(\tilde{\Sigma})(\tau, \tilde{\eta})\| \leqslant \kappa \|\eta - \tilde{\eta}\| + \nu \|\Sigma - \tilde{\Sigma}\| \|\eta\|, \quad (18)$$

in case that

$$\frac{\ell M^2(1 + \kappa)}{\gamma - \rho - 2\ell M(1 + \kappa)} \leqslant \kappa, \quad (19a)$$

$$\frac{2\ell M^2}{\gamma - \rho - 2\ell M(1 + \kappa)} < 1. \quad (19b)$$

Now, (19a) can be rewritten as $2M\kappa^2 + (M^2 + M - (\gamma - \rho)/\ell)\kappa + M^2 \leq 0$, which can be seen as a quadratic polynomial (in κ), admitting two real roots (due to (5)) given by

$$\kappa_{\pm} := \frac{\frac{\gamma-\rho}{\ell} - M^2 - 2M \pm \sqrt{(\frac{\gamma-\rho}{\ell} - M^2 - 2M)^2 - 8M^3}}{4M}. \quad (20)$$

Moreover, the condition $(\gamma - \rho)/\ell > M^2 + 2M + \sqrt{8M^3}$ in (5) implies that $(\gamma - \rho)/\ell > M^2 + 2M$ and thus $\kappa_+ > \kappa_- > 0$. Thus, (19a) is satisfied for any $\kappa \in [\kappa_-, \kappa_+]$.

Equation (19b) holds true for κ_- , due to $(\gamma - \rho)/\ell > 3M^2 + 2M$ in (5). Moreover, we can isolate κ in (19b), and thus this inequality is satisfied for any $\kappa < \kappa_* := (\gamma - \rho)/(2M\ell) - M - 1$.

Due to (5), $(\gamma - \rho)/\ell > 3M^2 + 2M$ and $\kappa_- < \kappa_*$. Therefore, both conditions (19) are satisfied for any $\kappa \in [\kappa_-, \min\{\kappa_+, \kappa_*\}]$.

Consequently, inequality (18) with $\Sigma = \tilde{\Sigma}$ implies that the image of the map G lies in $\mathcal{LB}_\Sigma(D)$ and, inequality (18) with $\eta = \tilde{\eta}$, shows that G is a contraction.

Therefore, the map G has a unique fixed point, $G(\Sigma^*) = \Sigma^*$. This establishes the existence of the invariant manifold and its invariance.

Furthermore, Σ^* being Lipschitz with constant $\kappa > 0$ and $\Sigma(t, 0) = 0$, together with (12), implies the growth estimate (8) within the invariant manifold.

This completes the first part of the proof.

We now embark in the second part of the proof.

For $(t, u) \in \mathbb{R} \times X$, define $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, u)$.

We show that $\mathcal{M}(t) = \{Im(P_{\Sigma^*}(t)) : t \in \mathbb{R}\}$ has the property that any solution satisfies (9) (exponential attraction if $\delta > 0$), and thus we wish to bound $\xi(t) := T(t, \tau)u - P_{\Sigma^*}(t)T(t, \tau)u$ for any $\eta \in X$ and $t \geq \tau$.

Note that $\xi(t) = p(t) - \Sigma^*(t, q(t))$ due to the definitions in (11).

Define $q^*(s, t)$, for $s \leq t$, as

$$q^*(s, t) := L(s, t)q(t) + \int_t^s L(s, r)Q(r)f(r, q^*(r, t) + \Sigma^*(r, q^*(r, t)))dr. \quad (21)$$

Since f, Σ^* are Lipschitz with constants $\ell, \kappa > 0$, for $s \leq t$,

$$\begin{aligned} & \|q^*(s, t) - q(s)\| \\ & \leq M \int_s^t e^{\rho(r-s)} \|f(r, q^*(r, t) + \Sigma^*(r, q^*(r, t))) - f(r, q(r) + p(r))\| dr \\ & \leq M\ell \int_s^t e^{\rho(r-s)} (\|\Sigma^*(r, q^*(r, t)) - p(r)\| + \|q^*(r, t) - q(r)\|) dr, \tag{22} \\ & \leq M\ell \int_s^t e^{\rho(r-s)} (\|\Sigma^*(r, q(r)) - p(r)\| + (1+\kappa) \|q^*(r, t) - q(r)\|) dr. \end{aligned}$$

Hence, by Gronwall's Lemma and definition of ξ ,

$$\|q^*(s, t) - q(s)\| \leq M\ell \int_s^t e^{(\rho + M\ell(1+\kappa))(r-s)} \|\xi(r)\| dr, \quad s \leq t. \quad (23)$$

Also, for $s \leq \tau \leq t$, we obtain

$$\begin{aligned} \|q^*(s, t) - q^*(s, \tau)\| &\leq \|L(s, \tau)Q(\tau)[q^*(\tau, t) - q(\tau)]\| \\ &+ \left\| \int_{\tau}^s L(s, r)Q(r)[f(r, q^*(r, t) + \Sigma^*(r, q^*(r, t))) - f(r, q^*(r, \tau) + \Sigma^*(r, q^*(r, \tau)))] dr \right\| \\ &\leq M^2 \ell e^{\rho(\tau-s)} \int_{\tau}^t e^{(\rho + M\ell(1+\kappa))(r-\tau)} \|\xi(r)\| dr \\ &+ M\ell(1 + \kappa) \int_s^{\tau} e^{\rho(r-s)} \|q^*(r, t) - q^*(r, \tau)\| dr, \end{aligned}$$

and by Grönwall's Lemma

$$\|q^*(s, t) - q^*(s, \tau)\| \leq M^2 \ell \int_{\tau}^t e^{(\rho + M\ell(1+\kappa))(r-s)} \|\xi(r)\| dr. \quad (24)$$

Now, we use these inequalities to estimate $\|\xi(t)\|$. Note that

$$\begin{aligned}
 \xi(t) - L(t, \tau)(I - Q(\tau))\xi(\tau) &= p(t) - L(t, \tau)p(\tau) - \Sigma^*(t, q(t)) + L(t, \tau)\Sigma^*(\tau, q(\tau)) \\
 &= \int_{\tau}^t L(t, s)(I - Q(s))f(s, q(s) + p(s))ds - \int_{-\infty}^t L(t, s)(I - Q(s))f(s, q^*(s, t)) + \Sigma^*(s, q^*(s, t)))ds \\
 &\quad + \int_{-\infty}^{\tau} L(t, s)(I - Q(s))f(s, q^*(s, \tau)) + \Sigma^*(s, q^*(s, \tau)))ds \\
 &= \int_{\tau}^t L(t, s)(I - Q(s))[f(s, q(s) + p(s)) - f(s, q^*(s, t)) + \Sigma^*(s, q^*(s, t))]ds \\
 &\quad - \int_{-\infty}^{\tau} L(t, s)(I - Q(s))[f(s, q^*(s, t)) + \Sigma^*(s, q^*(s, t)) - f(s, q^*(s, \tau)) + \Sigma^*(s, q^*(s, \tau))]ds.
 \end{aligned}$$

Thus, using (23) and (24), we obtain

$$\begin{aligned}
& \|\xi(t) - L(t, \tau)(I - Q(\tau))\xi(\tau)\| \\
& \leq M\ell \int_{\tau}^t e^{-\gamma(t-s)} (\|p(s) - \Sigma^*(s, q^*(s, t))\| + \|q(s) - q^*(s, t)\|) ds \\
& + M\ell(1 + \kappa) \int_{-\infty}^t e^{-\gamma(t-s)} \|q^*(s, \tau) - q^*(s, t)\| ds \\
& \leq M\ell \int_{\tau}^t e^{-\gamma(t-s)} \|\xi(s)\| ds \\
& + M^2\ell^2(1 + \kappa) \int_{\tau}^t e^{-\gamma(t-r)} \|\xi(r)\| \int_{\tau}^r e^{-(\gamma - \rho - M\ell(1 + \kappa))(r-s)} ds dr \\
& + M^3\ell^2(1 + \kappa) \int_{\tau}^t e^{-\gamma(t-r)} e^{-(\gamma - \rho - M\ell(1 + \kappa))(r-\tau)} \|\xi(r)\| \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)} e^{(\rho + M\ell(1 + \kappa))(\tau-s)} ds dr \\
& \leq M\ell \int_{\tau}^t e^{-\gamma(t-s)} \|\xi(s)\| ds + \frac{M^2\ell^2(1 + \kappa)}{\gamma - \rho - M\ell(1 + \kappa)} \int_{\tau}^t e^{-\gamma(t-r)} \|\xi(r)\| dr \\
& + \frac{M^3\ell^2(1 + \kappa)}{\gamma - \rho - M\ell(1 + \kappa)} \int_{\tau}^t \|\xi(r)\| e^{-\gamma(t-r)} dr
\end{aligned}$$

and we have that

$$\|\xi(t) - L(t, \tau)(I - Q(\tau))\xi(\tau)\| \leq \left[M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)} \right] \int_{\tau}^t e^{-\gamma(t-r)} \|\xi(r)\| dr.$$

Thus,

$$\|\xi(t)\| \leq Me^{-\gamma(t-\tau)}\|\xi(\tau)\| + \left[M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)} \right] \int_{\tau}^t e^{-\gamma(t-r)} \|\xi(r)\| dr$$

and

$$e^{\gamma t} \|\xi(t)\| \leq Me^{\gamma\tau}\|\xi(\tau)\| + \left[M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)} \right] \int_{\tau}^t e^{\gamma r} \|\xi(r)\| dr.$$

By Grönwall's Lemma, we obtain the bound in equation (9). \square

Teorema

Suppose that the linear evolution process $\{L(t, \tau) : t \geq \tau\}$ has exponential splitting, with constant $M \geq 1$, exponents $\gamma > \rho$ and a family of projections $\{Q(t) : t \in \mathbb{R}\}$.

If $(\gamma - \rho)/\ell$ satisfies (5), then there is a continuous function

$$\begin{aligned} \Theta^* : \mathbb{R} \times X &\rightarrow X \\ (t, u) &\mapsto \Theta^*(t, u), \end{aligned} \tag{25}$$

such that $\Theta^*(t, u) = \Theta^*(t, (I - Q(t))u) = Q(t)\Theta^*(t, u)$, and $\Theta^*(t, 0) = 0$ for all $t \in \mathbb{R}$, which is uniformly Lipschitz with constant $\kappa = \kappa(\gamma, \rho, \ell, M) > 0$, i.e.,

$$\|\Theta^*(t, u) - \Theta^*(t, \tilde{u})\| \leq \kappa \|u - \tilde{u}\| \text{ for all } (t, u), (t, \tilde{u}) \in \mathbb{R} \times X.$$

Moreover, if $P_{\Theta^*}(t)u := \Theta^*(t, (I - Q(t))u) + (I - Q(t))u$, for all $(t, u) \in \mathbb{R} \times X$, the family given by

$$\{Im(P_{\Theta^*}(t)) : t \in \mathbb{R}\} := \{\{P_{\Theta^*}^*(t, u) : u \in X\} : t \in \mathbb{R}\}, \quad (26)$$

is positively invariant such that

$$\|T(t, \tau)P_{\Theta^*}(\tau)u\| \leq M(1+\kappa)e^{-(\gamma-M\ell(1+\kappa))(t-\tau)}\|P_{\Theta^*}(\tau)u\|, \quad (27)$$

$t \geq \tau$, $u \in X$, and

$$\|u - P_{\Theta^*}(\tau)u\| \leq M e^{\hat{\delta}(t-\tau)} \|(I - P_{\Theta^*}(t))T(t, \tau)u\|, \quad (28)$$

$t \geq \tau$, $u \in X$, where $\hat{\delta} = \rho + M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)}$.

Furthermore, if $\gamma - M\ell(1 + \kappa) > 0$, $\{Im(P_{\Theta^*}(t)) : t \in \mathbb{R}\}$ is the stable manifold of the inertial manifold $\{Im(P_{\Sigma^*}(t)) : t \in \mathbb{R}\}$.

Proof: Given $\kappa > 0$ consider the complete metric space

$$\begin{aligned} \mathcal{LB}_\Theta(\kappa) = & \left\{ \Theta \in C(\mathbb{R} \times X, X) : \|\Theta(t, u) - \Theta(t, \tilde{u})\| \leq \kappa \|u - \tilde{u}\|, \Theta(t, 0) = 0, \right. \\ & \left. \Theta(t, u) = \Theta(t, (I - Q(t))u) \in \text{Im}(Q(t)), \forall (t, u), (t, \tilde{u}) \in \mathbb{R} \times X \right\}. \end{aligned} \quad (29)$$

with the metric $\|\Theta - \tilde{\Theta}\| = \sup_{t \in \mathbb{R}} \sup_{u \neq 0} \frac{\|\Theta(t, u) - \tilde{\Theta}(t, u)\|}{\|u\|}$.

Next we outline the heuristic procedure that will establish the way of proving that the invariant manifold is given as a graph of a map in $\mathcal{LB}_\Theta(\kappa)$. We are looking for $\Theta \in \mathcal{LB}_\Theta(\kappa)$ with the property that, if $(\tau, \eta) \in \mathbb{R} \times X$, then a solution u of (2), with initial data $u(\tau) = \Theta(\tau, (I - Q(\tau))\eta) + (I - Q(\tau))\eta \in X$, can be decomposed as $u(t) = q(t) + p(t)$, where $q(t) = \Theta(t, p(t))$ for all $t \geq \tau$. Thus, q and p must satisfy, for $t \geq \tau$,

$$q(t) = L(t, \tau)Q(\tau)\eta + \int_{\tau}^t L(t, s)Q(s)f(s, p(s) + \Theta(s, p(s)))ds, \quad (30a)$$

$$p(t) = L(t, \tau)(I - Q(\tau))\eta + \int_{\tau}^t L(t, s)(I - Q(s))f(s, p(s) + \Theta(s, p(s)))ds. \quad (30b)$$

It follows that

$$\begin{aligned} \|p(t)\| &\leq \|L(t, \tau)(I - Q(\tau))\eta\| \\ &+ \int_{\tau}^t \|L(t, s)(I - Q(s))f(s, p(s) + \Theta(s, p(s)))\|ds \\ &\leq M e^{-\gamma(t-\tau)}\|\eta\| + \int_{\tau}^t M \ell e^{-\gamma(t-s)}(1 + \kappa)\|p(s)\|ds. \end{aligned}$$

Using Grownwall's inequality,

$$\|p(t)\| \leq M e^{-(\gamma - M\ell(1+\kappa))(t-\tau)} \|\eta\|.$$

From this and from the fact that $q(t) = \Theta(p(t))$, we conclude that

$$\begin{aligned} \|L(\tau, t)Q(t)q(t)\| &= \|L(\tau, t)\Theta(p(t))\| \\ &\leq \kappa M^2 e^{-(\gamma - \rho - M\ell(1+\kappa))(t-\tau)} \|\eta\|. \end{aligned}$$

Applying $L(\tau, t)Q(t)$ to (30), using that

$\Theta(\tau, (I - Q(\tau))\eta) = Q(\tau)\eta$ and making $t \rightarrow \infty$ we have

$$0 = \Theta(\tau, (I - Q(\tau))\eta) + \int_{\tau}^{\infty} L(\tau, s)Q(s)f(s, p(s) + \Theta(s, p(s)))ds.$$

Inspired by this we define the operator $\tilde{G} : \mathcal{LB}_{\Theta}(\kappa) \rightarrow \mathcal{LB}_{\Theta}(\kappa)$ by

$$\tilde{G}(\Theta)(\tau, \eta) = - \int_{\tau}^{\infty} L(\tau, s)Q(s)f(s, p(s) + \Theta(s, y(s)))ds, \quad (31)$$

$(\tau, \eta) \in \mathbb{R} \times X$.

The fact that \tilde{G} is a well-defined contraction is similar to Theorem 1, and we refrain from giving a proof. Hence \tilde{G} admits a unique fixed point $\Theta^* \in \mathcal{LB}_\Theta(\kappa)$ satisfying the desired properties.

We now embark in the proof of (28). For any $(\tau, \eta) \in X$ and $t \geq \tau$.

$$p(t) = L(t, \tau)(I - Q(\tau))\eta + \int_{\tau}^t L(t, s)(I - Q(s))f(s, q(s) + p(s))ds$$

and thus we wish to bound the variable

$\eta(t) := T(t, \tau)u - P_{\Theta^*}(t)T(t, \tau)u$ for any $u \in X$ and $t \geq \tau$. Note that $\eta(t) = q(t) - \Theta^*(t, p(t))$ due to the definitions in (30).

Define $p^*(s, t)$, for $s \geq t$, as

$$\begin{aligned} p^*(s, t) := & L(s, t)p(t) \\ & + \int_t^s L(s, r)(I - Q(r))f(r, \Theta^*(r, p^*(r, t)) + p^*(r, t))dr. \end{aligned} \quad (32)$$

Since f, Θ^* are Lipschitz with respective constants $\ell, \kappa > 0$, we obtain

$$\begin{aligned} & \|p^*(s, t) - p(s)\| \\ & \leq M \int_t^s e^{-\gamma(s-r)} \|f(r, \Theta^*(r, p^*(r, t)) + p^*(r, t)) - f(r, q(r) + p(r))\| dr \end{aligned} \quad (33)$$

$$\leq M\ell \int_t^s e^{-\gamma(s-r)} (\|q(r) - \Theta^*(r, p(r))\| + (1+\kappa)\|p^*(r, t) - p(r)\|) dr,$$

and, by Grönwall's Lemma,

$$\|p^*(s, t) - p(s)\| \leq M\ell \int_t^s e^{-(\gamma - M\ell(1+\kappa))(s-r)} \|\eta(r)\| dr. \quad (34)$$

Also, for $s \geq t \geq \tau$, we obtain

$$\begin{aligned}
 & \|p^*(s, \tau) - p^*(s, t)\| \leq \|L(s, t)(I - Q(t))[p^*(t, \tau) - p(t)]\| \\
 & + \left\| \int_t^s L(s, r)(I - Q(r)) [f(r, \Theta^*(r, p^* v(r, \tau)) + p^*(r, \tau)) - f(r, \Theta^*(r, p^*(r, t)) + p^*(r, t))] dr \right\| \\
 & \leq M^2 \ell e^{-\gamma(s-t)} \int_{\tau}^t e^{-(\gamma - M\ell(1+\kappa))(t-r)} \|\eta(r)\| dr \\
 & + M\ell(1+\kappa) \int_t^s e^{-\gamma(s-r)} \|p^*(r, \tau) - p^*(r, t)\| dr,
 \end{aligned}$$

and again by Grönwall's Lemma,

$$\|p^*(s, \tau) - p^*(s, t)\| \leq M^2 \ell \int_{\tau}^t e^{-(\gamma - M\ell(1+\kappa))(s-r)} \|\eta(r)\| dr. \quad (35)$$

Now, we use these inequalities to estimate $\|\eta(\tau)\|$. Note that

$$\begin{aligned}\eta(\tau) - L(\tau, t)Q(t)\eta(t) &= q(\tau) - L(\tau, t)q(t) - \Theta^*(\tau, p(\tau)) + L(\tau, t)\Theta^*(t, p(t)) \\ &= \int_t^\tau L(\tau, s)Q(s)[f(s, q(s) + p(s)) - f(s, \Theta^*(s, p^*(s, \tau)) + p^*(s, \tau))]ds \\ &\quad + \int_t^\infty L(\tau, s)Q(s)[f(s, \Theta^*(s, p^*(s, \tau)) + p^*(s, \tau)) - f(s, \Theta^*(s, p^*(s, t)) + p^*(s, t))]ds.\end{aligned}$$

Thus, using (34) and (35), we obtain

$$\begin{aligned}
 & \| \eta(\tau) - L(\tau, t)Q(t)\eta(t) \| \\
 & \leq M\ell \int_{\tau}^t e^{-\rho(\tau-s)} (\| q(s) - \Theta^*(s, p^*(s, \tau)) \| + \| p(s) - p^*(s, \tau) \|) ds \\
 & + M\ell(1 + \kappa) \int_t^{\infty} e^{-\rho(\tau-s)} \| p^*(s, t) - p^*(s, \tau) \| ds \\
 & \leq M\ell \int_{\tau}^t e^{-\rho(\tau-s)} \| \eta(s) \| ds \\
 & + M^2 \ell^2 (1 + \kappa) \int_{\tau}^t e^{-(\gamma - \rho - M\ell(1 + \kappa))(s - \tau)} \int_{\tau}^s e^{-(\gamma - M\ell(1 + \kappa))(\tau - r)} \| \eta(r) \| dr ds \\
 & + M^3 \ell^2 (1 + \kappa) \int_t^{\infty} e^{-(\gamma - \rho - M\ell(1 + \kappa))(s - \tau)} \int_{\tau}^t e^{-(\gamma - M\ell(1 + \kappa))(\tau - r)} \| \eta(r) \| dr ds.
 \end{aligned}$$

Hence

$$\begin{aligned} \|\eta(\tau) - L(\tau, t)Q(t)\eta(t)\| &\leq M\ell \int_{\tau}^t e^{-\rho(\tau-s)} \|\eta(s)\| ds \\ &+ \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)} \int_{\tau}^t e^{-(\gamma-\rho-M\ell(1+\kappa))(\tau-r)} e^{-\rho(\tau-r)} \|\eta(r)\| dr \end{aligned}$$

and we have that

$$\|\eta(\tau) - L(\tau, t)Q(t)\eta(t)\| \leq \left[M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)} \right] \int_{\tau}^t e^{-\rho(\tau-r)} \|\eta(r)\| dr.$$

Thus,

$$\|\eta(\tau)\| \leq M e^{-\rho(\tau-t)} \|\eta(t)\| + \left[M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-M\ell(1+\kappa)} \right] \int_{\tau}^t e^{-\rho(\tau-r)} \|\eta(r)\| dr.$$

By Grönwall's Lemma, we obtain the bound in (28). \square