SMA 5878 Functional Analysis II

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Spectral decomposition: $A \in \mathcal{K}(H)$ self-adjoint

What I had in mind was the following
Take two functions $h(t,x)$ and $g(t,x)$ for which you can prove that the driving has a finite dimensional attractor.
Now connect them from $-\infty$ to $+\infty$ by a function like $\frac{e^t}{1+e^t}$, that is, consider

$$f(t,x) = (1 - r(t))h(t,x) + r(t)g(t,x), \quad t \in (-\infty, \infty)$$

The new driving will have an attractor and you can control the dimension of the new driving in terms of the dimensions of the drivings associated to h and g using the results with Matheus.
Please note that the transition between h and g is exponential and should not affect the dimension much.
We can talk more over Skype.
Let H be a Hilbert space over $\mathbb{C}$. As a consequence of a previous result we have the following:

Proof: First note that $R(P_{\{\lambda_0\}}(A)) \neq \{0\}$ (since $\sigma(\mathbf{A}_{\{\lambda_0\}}) = \{\lambda_0\} \neq \varnothing$). From the fact that $\sigma(\lambda_0 - \mathbf{A}_{\{\lambda_0\}}) = \{0\}$ and from a previous result it follows that $\mathbf{A}_{\{\lambda_0\}} = \lambda_0 I_{H_{\{\lambda_0\}}}$. From that it follows that λ_0 is an eigenvalue of A and that $N(\lambda_0 - A) \supset R(P_{\{\lambda_0\}})$.

On the other hand, if $x \in N(\lambda_0 - A)$, $r > 0$ is such that $\overline{B}_r(\lambda_0)\backslash\{\lambda_0\} \subset \rho(A)$, $\gamma(t) = \lambda_0 + re^{2\pi i t}$, $t \in [0,1]$,

$$P_{\{\lambda_0\}}(A)x = \frac{1}{2\pi i}\int_\gamma (\lambda - A)^{-1} d\lambda\, x = x,$$

where we have used that $(\lambda - A)^{-1} = \dfrac{I - (\lambda_0 - A)(\lambda - A)^{-1}}{\lambda - \lambda_0}$.

Hence $x \in R(P_{\{\lambda_0\}})$ showing that $R(P_{\{\lambda_0\}}) = N(\lambda_0 - A)$.$\square$

Exercise

Let A be a self-adjoint operator. If σ is a spectral set of A, show that P_σ is self-adjoint and conclude that P_σ is an orthogonal projection.

If $A : H \to H$ is a **compact self-adjoint operator**. It follows from what we have previously seen, all points in $\sigma(A)\backslash\{0\}$ are isolated eigenvalues of finite multiplicity.

If $\sigma(A)\backslash\{0\} = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$, we define $P_n = P_{\{\lambda_n\}}$ and P_0 an orthogonal projection with image $N(A)$.

If Y is the subspace of H generated by $\cup_{n=0}^{\infty} P_n H$, let us show that Y is dense in H.

It is clear that $AY \subset Y$ and $AY^{\perp} \subset Y^{\perp}$ and if $A_0 = A_{|Y^{\perp}}$, then A_0 is compact self-adjoint. If $Y^{\perp} \neq \{0\}$, then $\sigma(A_0) = \{0\}$, $A_0 = 0$ and $Y^{\perp} \subset N(A) = R(P_0)$ which is a contradiction.

It follows that, for all $x \in H$,

$$x = \sum_{n=0}^{\infty} P_n x$$

and

$$Ax = \sum_{n=1}^{\infty} \lambda_n P_n x$$

with the series converging in H.

Now if $A : D(A) \subset H \to H$ is a self-adjoint operator with **compact resolvent**.

It follows that $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$ and, if $Y = \cup_{j=1}^{\infty} R(P_{\{\lambda_j\}})$ then, $Y^{\perp} \cap D(A) = \{0\}$ for, otherwise, the restriction A_0 of A to $Y^{\perp}$ would be a self-adjoint operator with compact resolvent such that $0 \in \rho(A_0)$ and $A_0^{-1} = 0$ (since $\rho(A_0) = \mathbb{C}$ and consequently $\sigma(A_0^{-1}) = \{0\}$) which results $R(A_0^{-1}) = D(A_0) = \{0\}$ and we have a contradition.

Hence, if $x \in D(A)$

$$x = \sum_{j=1}^{\infty} P_j x$$

and

$$Ax = \sum_{j=1}^{\infty} \lambda_j P_j x.$$

Continuity of the spectrum

Let X_ϵ be a family of Banach spaces, $\epsilon \in [0,1]$, and suppose that there is a family of bounded linear operators $E_\epsilon : X \to X_\epsilon$ such that

$$\|E_\epsilon u\|_{X_\epsilon} \xrightarrow{\epsilon \to 0} \|u\|_X, \qquad \text{for all } u \in X. \tag{1}$$

Definition

We say that a family $\{u_\epsilon\}_{\epsilon \in (0,1]}$, with $u_\epsilon \in X_\epsilon$ for all $\epsilon \in [0,1]$, $E-$converges to u if $\|u_\epsilon - E_\epsilon u\|_{X_\epsilon} \xrightarrow{\epsilon \to 0} 0$. We write $u_\epsilon \xrightarrow{E} u$ to say that $\{u_\epsilon\}_{\epsilon \in [0,1]}$ E-converges to u as ϵ tends to zero.

With this notion of convergence we present the definition of E-relatively compact sequence.

Definition

A sequence $\{u_n\}_{n\in\mathbb{N}}$, with $u_n \in X_{\epsilon_n}$ and $\epsilon_n \to 0$, is said E-relatively compact if, for each subsequence $\{u_{n'}\}$ of $\{u_n\}$, there is a subsequence $\{u_{n''}\}$ of $\{u_{n'}\}$ and an element $u \in X$ such that $u_{n''} \xrightarrow{E} u$. A family $\{u_\epsilon\}_{\epsilon\in(0,1]}$ is said E-relatively compact if each sequence $\{u_{\epsilon_n}\}$, $\epsilon_n \to 0$, is E-relatively compact.

Definition

We say that a family of operators $\{B_\epsilon \in \mathcal{L}(X_\epsilon)\}_{\epsilon\in[0,1]}$ EE-converges to B_0 as $\epsilon \to 0$, if $B_\epsilon u_\epsilon \xrightarrow{E} B_0 u$ whenever $u_\epsilon \xrightarrow{E} u \in X$. We write $B_\epsilon \xrightarrow{EE} B_0$ to denote that $\{B_\epsilon \in \mathcal{L}(X_\epsilon)\}_{\epsilon\in[0,1]}$ EE-converges to B_0 as $\epsilon \to 0$.

Definition

We say that a family of compact operators $\{B_\epsilon \in \mathcal{K}(X_\epsilon) : \epsilon \in [0,1]\}$ *compactly converges to* B_0 *as* $\epsilon \to 0$ *if* $B_\epsilon \xrightarrow{EE} B_0$ *and for each family* $\{u_\epsilon\}$ *with* $u_\epsilon \in X_\epsilon$, $\|u_\epsilon\|_{X_\epsilon} = 1$, $\epsilon \in (0,1]$, *the family* $\{B_\epsilon u_\epsilon\}_{\epsilon \in [0,1]}$ *is* E*-relatively compact. We write* $B_\epsilon \xrightarrow{CC} B_0$ *to denote that* $\{B_\epsilon \in \mathcal{K}(X_\epsilon) : \epsilon \in [0,1]\}$ *compactly converges to* B_0 *as* $\epsilon \to 0$.

Our next result plays a fundamental role in the proofs of the main results of this section.

Lemma

Let $\{B_\epsilon \in \mathcal{K}(X_\epsilon)\}_{\epsilon\in[0,1]}$ *be such that* $B_\epsilon \xrightarrow{CC} B_0$. *Then,*

i) there is an $\epsilon_0 \in (0, \epsilon]$ *such that* $\sup_{\epsilon\in(0,\epsilon_0]} \|B_\epsilon\|_{\mathcal{L}(X_\epsilon)} < \infty$.

ii) if $\mathcal{N}(I + B_0) = \{0\}$, *there is an* $\epsilon_0 > 0$ *and* $M > 0$ *such that* $\mathcal{N}(I + B_\epsilon) = \{0\}$ *for all* $\epsilon \in [0, \epsilon_0]$ *and*

$$\|(I + B_\epsilon)^{-1}\|_{\mathcal{L}(X_\epsilon)} \leqslant M, \quad \forall \epsilon \in [0, \epsilon_0]. \tag{2}$$

Proof:

$i)$ If $\{\|B_\epsilon\|_{\mathcal{L}(X_\epsilon)} : \epsilon \in (0, \epsilon_0]\}$ is not bounded for any choice of $\epsilon_0 \in (0, 1]$, there is a sequence $\{\epsilon_n\}$ in $(0, 1]$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$ and $u_{\epsilon_n} \in X_{\epsilon_n}$ with $\|u_{\epsilon_n}\|_{X_{\epsilon_n}} = 1$ such that $\|B_{\epsilon_n} u_{\epsilon_n}\| \to +\infty$.

This is in contradiction with the compact convergence of B_ϵ to B_0.

ii) First assume that there are no $\epsilon_0 > 0$ such that $\mathcal{N}(I + B_\epsilon) = \{0\}$ for all $\epsilon \in [0, \epsilon_0]$.

In this case, there are sequences $\{\epsilon_n\}$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$ and $u_n \in X_{\epsilon_n}$ with $\|u_n\|_{X_{\epsilon_n}} = 1$ such that $u_n + B_{\epsilon_n} u_n = 0$.

Since $B_\epsilon \overset{CC}{\longrightarrow} B_0$, we may assume (taking a subsequence if necessary) that $B_{\epsilon_n} u_n \overset{E}{\longrightarrow} -u$ with $\|u\|_X = 1$ and consequently $u_n \overset{E}{\longrightarrow} u$.

It follows that $u + B_0 u = 0$ and this is in contradiction with $\mathcal{N}(I + B_0) = \{0\}$.

Now, let us prove (2). Since $B_\epsilon \in \mathcal{K}(X_\epsilon)$ for each $\epsilon \in [0,1]$, It follows from Fredholm Alternative (see Theorem 6.6.c. in [Brezis]) that the estimate (2) is equivalent to

$$\|(I + B_\epsilon)u_\epsilon\|_{X_\epsilon} \geqslant \frac{1}{M}, \quad \forall \epsilon \in [0, \epsilon_0] \text{ and } \forall u_\epsilon \in X_\epsilon \text{ with } \|u_\epsilon\|_{X_\epsilon} = 1.$$

Suppose not, that is, suppose that there is a sequence $\{u_n\}$, with $u_n \in X_{\epsilon_n}$, $\|u_n\| = 1$ and $\epsilon_n \to 0$ such that $\|(I + B_{\epsilon_n})u_n\| \to 0$.

Since $\{B_{\epsilon_n} u_n\}$ has an E-convergente subsequence, that we again denote by $\{B_{\epsilon_n} u_n\}$, to u, $\|u\| = 1$, it follows that $u_n + B_{\epsilon_n} u_n \xrightarrow{E} 0$ and $u_n \xrightarrow{E} -u$.

This implies that $(I + B_0)u = 0$ which is in contradiction with $N(I + B_0) = \{0\}$.$\square$

In general, the operators B_ϵ are the inverses of differential operators A_ϵ.

So, suppose that $\{A_\epsilon : D(A_\epsilon) \subset X_\epsilon \to X_\epsilon,\ \epsilon \in [0,1]\}$ is a family of closed operators and that, for all $\epsilon \in [0,1]$

$$\boxed{A_\epsilon \ \text{ has compact resolvente, } 0 \in \rho(A_\epsilon) \text{ and } A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}}. \quad (3)$$

Lemma

Assume that $\{A_\epsilon : D(A_\epsilon) \subset X_\epsilon \to X_\epsilon,\ \epsilon \in [0,1]\}$ *satisfies* (3).

Then, for each $\lambda \in \rho(A_0)$*, there is an* $\epsilon_\lambda > 0$ *such that* $\lambda \in \rho(A_\epsilon)$ *for all* $\epsilon \in [0, \epsilon_\lambda]$ *and there is a constant* $M_\lambda > 0$ *such that*

$$\|(\lambda - A_\epsilon)^{-1}\| \leqslant M_\lambda, \quad \forall \epsilon \in [0, \epsilon_\lambda]. \tag{4}$$

Besides that, $(\lambda - A_\epsilon)^{-1} \xrightarrow{CC} (\lambda - A_0)^{-1}$ *as* $\epsilon \to 0$.

Proof: From (3) and from the fact that $\lambda \in \rho(A_0)$ it is easy to see that $(\lambda - A_0)^{-1} = -A_0^{-1}(I - \lambda A_0^{-1})^{-1}$.

Since $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$, applying Lemma 1 i) and ii), we obtain that the operator $-A_\epsilon^{-1}(I - \lambda A_\epsilon^{-1})^{-1}$ is well defined and is bounded.

Simple calculations show that $-A_\epsilon^{-1}(I - \lambda A_\epsilon^{-1})^{-1} = (\lambda - A_\epsilon)^{-1}$. Hence $\lambda \in \rho(A_\epsilon)$ and we obtain (4).

To prove the compact convergence of $(\lambda - A_\epsilon)^{-1}$ to $(\lambda - A_0)^{-1}$ we proceed in the following manner:

Since A_ϵ^{-1} compactly converges to A_0^{-1} and since $\{(I - \lambda A_\epsilon^{-1})^{-1} : 0 \leqslant \epsilon \leqslant \epsilon_\lambda\}$ is bounded we conclude that

- If $\|u_\epsilon\|_{X_\epsilon} = 1$ then $(\lambda - A_\epsilon)^{-1} u_\epsilon = -A_\epsilon^{-1} w_\epsilon$ with $w_\epsilon = (I - \lambda A_\epsilon^{-1})^{-1} u_\epsilon$ which is bounded uniformly in ϵ. Hence $(\lambda - A_\epsilon)^{-1} u_\epsilon$ has an E-convergent subsequence.

- If $u_\epsilon \xrightarrow{E} u$ then $A_\epsilon^{-1} u_\epsilon \xrightarrow{E} A_0^{-1} u$. Now, for each subsequence of $\{(\lambda - A_\epsilon)^{-1} u_\epsilon\}$ there is a subsequence (which we again denote by $\{(\lambda - A_\epsilon)^{-1} u_\epsilon\}$) and a $y \in X$ such that,

$$\begin{aligned}(\lambda - A_\epsilon)^{-1} u_\epsilon &= -(I - \lambda A_\epsilon^{-1})^{-1} A_\epsilon^{-1} u_\epsilon \\ &= -A_\epsilon^{-1}(I - \lambda A_\epsilon^{-1})^{-1} u_\epsilon = z_\epsilon \xrightarrow{E} y.\end{aligned}$$

- Thus,

$$A_0^{-1} u \xleftarrow{E} A_\epsilon^{-1} u_\epsilon = -(I - \lambda A_\epsilon^{-1}) z_\epsilon \xrightarrow{E} -(I - \lambda A_0^{-1}) y$$

which gives $y = (\lambda - A_0)^{-1} u$.

- In particular, y is independente of the subsequence taken. This implies that the entire sequence $(\lambda - A_\epsilon)^{-1} u_\epsilon$ E-converges to $y = (\lambda - A_0)^{-1} u$ as $\epsilon \to 0$. Therefore, $(\lambda - A_\epsilon)^{-1} \xrightarrow{EE} (\lambda - A_0)^{-1}$ as $\epsilon \to 0$.

- From this it follows that $(\lambda - A_\epsilon)^{-1} \xrightarrow{CC} (\lambda - A_0)^{-1}$ as $\epsilon \to 0$ and the result is proved.$\square$

Lemma

Suppose that $\{A_\epsilon : D(A_\epsilon) \subset X_\epsilon \to X_\epsilon,\ \epsilon \in [0,1]\}$ *satisfies* (3). *If* Σ *is a compact subset of* $\rho(A_0)$, *there exists* $\epsilon_\Sigma > 0$ *such that* $\Sigma \subset \rho(A_\epsilon)$ *for all* $\epsilon \leqslant \epsilon_\Sigma$ *and*

$$\sup_{\epsilon \in [0,\epsilon_\Sigma]} \sup_{\lambda \in \Sigma} \|(\lambda - A_\epsilon)^{-1}\|_{\mathcal{L}(X_\epsilon)} < \infty. \tag{5}$$

Besides that, for each $u \in X$ *we have that*

$$\sup_{\lambda \in \Sigma} \|(\lambda - A_\epsilon)^{-1} E_\epsilon u - E_\epsilon (\lambda - A_0)^{-1} u\|_{X_\epsilon} \xrightarrow{\epsilon \to 0} 0. \tag{6}$$

Proof: First we show that there is a $\hat{\epsilon}_\Sigma > 0$ such that $\Sigma \subset \rho(A_\epsilon)$ for all $\epsilon \in [0, \hat{\epsilon}_\Sigma)$.

If that were not the case there would be subsequences $\epsilon_n \to 0$, $\lambda_n \in \Sigma$ (that we may suppose convergent to a $\lambda \in \Sigma$) and $u_{\epsilon_n} \in X_{\epsilon_n}$, $\|u_{\epsilon_n}\| = 1$ such that $A_{\epsilon_n} u_{\epsilon_n} - \lambda_n u_{\epsilon_n} = 0$ or equivalently $\lambda_n (A_{\epsilon_n})^{-1} u_{\epsilon_n} = u_{\epsilon_n}$.

From the compact convergence $\{u_{\epsilon_n}\}$ has an E-convergent subsequence to $u \in X$, $\|u\|_X = 1$ and $A_0 u = \lambda u$ which is in contradiction with $\sigma(A_0) \cap \Sigma = \emptyset$.

Let us show that there is an $\epsilon_\Sigma \in (0, \hat{\epsilon}_\Sigma)$ such that (5) holds. It is enough to prove that there is an $\epsilon_\Sigma \in (0, 1]$ such that

$$\{\|(I - \lambda A_\epsilon^{-1})^{-1}\|_{\mathcal{L}(X_\epsilon)} : \epsilon \in [0, \epsilon_\Sigma] \text{ and } \lambda \in \Sigma\} \text{ is bounded.}$$

If that was not the case there would be a sequence $\{\lambda_n\}$ in Σ (that we can suppose convergent to a certain $\tilde{\lambda} \in \Sigma$) and a sequence $\{\epsilon_n\}$ in $(0, 1]$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$ such that

$$\|(I - \lambda_n (A_{\epsilon_n})^{-1})^{-1}\|_{\mathcal{L}(X_{\epsilon_n})} \overset{n\to\infty}{\longrightarrow} \infty$$

From Lemma 1, since $-\lambda_n(A_{\epsilon_n})^{-1} \xrightarrow{CC} -\tilde{\lambda}(A_0)^{-1}$, we have a contradiction.

We will also prove (6) by contradiction. Suppose that there are sequences $\epsilon_n \to 0$, $\Sigma \ni \lambda_n \to \bar{\lambda} \in \Sigma$, $u \in X$ and $\eta > 0$ such that

$$\|(\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - E_{\epsilon_n}(\lambda_n - A_0)^{-1} u\|_{X_{\epsilon_n}} \geqslant \eta. \tag{7}$$

Using the resolvent identity, we have that

$$(\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - (\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u$$
$$= (\bar{\lambda} - \lambda_n)(\lambda_n - A_{\epsilon_n})^{-1} (\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u.$$

From that and (5) it follows that

$$\|(\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - (\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u\|_{X_{\epsilon_n}} \overset{n \to \infty}{\longrightarrow} 0. \tag{8}$$

From Lemma 2 we have that

$$\|(\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - E_{\epsilon_n} (\bar{\lambda} - A_0)^{-1} u\|_{X_{\epsilon_n}} \overset{n\to\infty}{\longrightarrow} 0. \tag{9}$$

Finally, from the continuity of the resolvent we have that

$$\|(\lambda_n - A_0)^{-1} u - (\bar{\lambda} - A_0)^{-1} u\|_X \overset{n\to\infty}{\longrightarrow} 0. \tag{10}$$

Now, (8), (9) and (10) are in contradiction with (7) and the result is proved. $\square$

For each $\delta > 0$ and $\lambda_0 \in \mathbb{C}$ define $S_\delta(\lambda_0) := \{\mu \in \mathbb{C} : |\lambda - \lambda_0| = \delta\}$.

To an isolated point $\lambda \in \sigma(A_0)$ we associate its generalized eigenspace $W(\lambda, A_0) = Q(\lambda, A_0)X$ where

$$Q(\lambda, A_0) = \frac{1}{2\pi i} \int_{|\xi - \lambda| = \delta} (\xi I - A_0)^{-1} d\xi$$

and δ is chosen in such a way that there is no other element of $\sigma(A_0)$ in the disc $\overline{B}_\delta^{\mathbb{C}}(\lambda) = \{\xi \in \mathbb{C} : |\xi - \lambda| \leqslant \delta\}$.

It follows from Lemma 3 that there is an $\epsilon_{S_\delta(\lambda)}$ such that $\rho(A_\epsilon) \supset S_\delta(\lambda)$ for all $\epsilon \leqslant \epsilon_{S_\delta(\lambda)}$. Let $W(\lambda, A_\epsilon) := Q(\lambda, A_\epsilon)X_\epsilon$ where

$$Q(\lambda, A_\epsilon) = \frac{1}{2\pi i} \int_{|\xi-\lambda|=\delta} (\xi I - A_\epsilon)^{-1} d\xi.$$

Exercise

Let X be a Banach space. If M, N are subspaces of X with $\dim(M) > \dim(N)$, show that there is a $u \in M$, $\|u\| = 1$ such that $\operatorname{dist}(u, N) = 1$ (Lemma IV.2.3 em [Kato-Perturbation Theory]).

Exercise

Let X be a Banach space. Show that, if P and Q are projections and $\dim(R(P)) > \dim(R(Q))$, then $\|P - Q\|_{\mathcal{L}(X)} \geq 1$.

The following result says that the spectrum of A_ϵ *approaches* the spectrum of A_0 as ϵ tends to zero.

We already know that the spectrum of A_ϵ or A_0 contains only isolated eigenvalues of finite multiplicity.

Theorem

Let $\{A_\epsilon : D(A_\epsilon) \subset X_\epsilon \to X_\epsilon,\ \epsilon \in [0,1]\}$ be a family of operators satisfying (3). Then, the following statements hold:

(i) *If $\lambda_0 \in \sigma(A_0)$, there is a sequence $\{\epsilon_n\}$ in $(0,1]$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$ and a sequence $\{\lambda_n\}$ in $\mathbb{C}$ with $\lambda_n \in \sigma(A_{\epsilon_n})$, for $n = 1, 2, 3 \cdots$, and $\lambda_n \overset{n\to\infty}{\longrightarrow} \lambda_0$.*

(ii) *If $\{\epsilon_n\}$ is a sequence in $(0,1]$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$, and $\{\lambda_n\}$ is a sequence in $\mathbb{C}$ with $\lambda_n \in \sigma(A_{\epsilon_n})$, $n \in \mathbb{N}$ and $\lambda_n \overset{n\to\infty}{\longrightarrow} \lambda_0$, then $\lambda_0 \in \sigma(A_0)$.*

(iii) *If $\lambda_0 \in \sigma(A_0)$, there is an $\epsilon_1 \in (0,1]$ such that $\dim W(\lambda_0, A_\epsilon) = \dim W(\lambda_0, A_0)$ for all $0 \leqslant \epsilon \leqslant \epsilon_1$.*

(iv) *If $u \in W(\lambda_0, A_0)$, there is a sequence $\{\epsilon_n\}$ in $(0,1]$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$, $u_{\epsilon_n} \in W(\lambda_0, A_{\epsilon_n})$ and such that $u_{\epsilon_n} \overset{E}{\longrightarrow} u$ as $n \to \infty$.*

(v) *If $\{\epsilon_n\}$ is a sequence in $(0,1]$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$, and $\{u_n\}$ is a sequence with $u_n \in W(\lambda_0, A_{\epsilon_n})$, $\|u_n\|_{X_{\epsilon_n}} = 1$, then $\{u_n\}$ has an $E-$convergent subsequence to a vector u in $W(\lambda_0, A_0)$.*

Proof:

(i) Let $\lambda_0 \in \sigma(A_0)$ and $\delta_0 > 0$ such that $\overline{B}^{\mathbb{C}}_{\delta_0}(\lambda_0) \cap \sigma(A_0) = \{\lambda_0\}$.

From a previous result there is and $\epsilon_0 > 0$ such that $S_{\delta_0}(\lambda_0) \subset \rho(A_\epsilon)$ for $\epsilon \in [0, \epsilon_0]$ and $\{\|(\lambda - A_\epsilon)^{-1}\|_{\mathcal{L}(X_\epsilon)} : \epsilon \in [0, \epsilon_0]$ and $\lambda \in S_{\delta_0}(\lambda_0)\}$ is bounded.

Now suppose that there is a $0 < \delta < \delta_0$ and a sequence $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$ such that, $\overline{B}_\delta(\lambda_0) \subset \rho(A_{\epsilon_n})$ for all $n \in \mathbb{N}$.

Since $\overline{B}_\delta(\lambda_0) \ni \lambda \mapsto (\lambda - A_{\epsilon_n})^{-1} \in \mathcal{L}(X_{\epsilon_n})$ is analytic for each $n \in \mathbb{N}$, from the proof of the previous lemma and from the Maximum Modulus Theorem we have that

$$\|(I - \lambda_0 A_{\epsilon_n}^{-1})^{-1}\|_{\mathcal{L}(X_{\epsilon_n})} \leqslant \sup_{\substack{|\lambda-\lambda_0|=\delta \\ n\in\mathbb{N}}} \|(I - \lambda A_{\epsilon_n}^{-1})^{-1}\|_{\mathcal{L}(X_{\epsilon_n})} < \infty.$$

Therefore, if $u \in X$, it follows that

$$\|(\lambda_0 A_0^{-1} - I)u\|_X = \lim_{n\to\infty} \|(\lambda_0 A_{\epsilon_n}^{-1} - I)E_{\epsilon_n} u\|_{X_{\epsilon_n}} \geqslant c\|u\|_X,$$

for some $c > 0$ and consequently $\lambda_0 \in \rho(A_0)$.

This contradicts the choice of λ_0 and proves that, for each $\delta > 0$, $\overline{B}_\delta(\lambda_0)$ contains some point of $\sigma(A_\epsilon)$, for all sufficiently small ϵ.

(ii) Let $\{\epsilon_n\}$ be a sequence in $(0,1]$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$, $\{\lambda_n\}$ a sequence in $\mathbb{C}$ with $\lambda_n \in \sigma(A_{\epsilon_n})$ such that $\lambda_n \overset{n\to\infty}{\longrightarrow} \lambda$ and $\{u_n\}$ a sequence with $u_n \in X_{\epsilon_n}$, $(I - \lambda_n(A_{\epsilon_n})^{-1})u_n = 0$ and $\|u_n\| = 1$.

Then

$$\|(I-\lambda(A_{\epsilon_n})^{-1})u_n\|_{X_{\epsilon_n}} = \|(I-\lambda_n(A_{\epsilon_n})^{-1})u_n - (\lambda-\lambda_n)(A_{\epsilon_n})^{-1}u_n\|_{X_{\epsilon_n}} \to 0$$

as $n \to \infty$.

Since $\|u_n\| = 1$ we have, taking subsequences if necessary, $\lambda(A_{\epsilon_n})^{-1}u_n \xrightarrow{E} u$ and $u_n \xrightarrow{E} u$ with $\|u\| = 1$. Therefore $u - \lambda A_0^{-1}u = 0$, $u \neq 0$ and $\lambda \in \sigma(A_0)$.

(iii) Since $(\lambda - A_\epsilon)^{-1} \xrightarrow{EE} (\lambda - A_0)^{-1}$ uniformly for $\lambda \in S_\delta(\lambda_0)$ (see previous lemma) it follows that $Q_\epsilon(\lambda_0) \xrightarrow{EE} Q(\lambda_0)$ as $\epsilon \to 0$.

If $v_1, \cdots, v_k$ is a basis for $W(\lambda_0, A_0) = Q_0(\lambda_0)X$, it is easy to see that, for all sufficiently small ϵ

$$\{Q_\epsilon(\lambda_0)E_\epsilon v_1, \cdots, Q_\epsilon(\lambda_0)E_\epsilon v_k\}$$

is a linearly independent set in $Q_\epsilon(\lambda_0)X_\epsilon$.

From this it follows that $\dim(Q_\epsilon(\lambda_0)(X_\epsilon)) \geqslant \dim(Q(\lambda_0)(X))$.

We prove the equality supposing that $Q_\epsilon(\lambda_0) \xrightarrow{CC} Q(\lambda_0)$.

By reduction to absurdum we assume that, for some sequence $\epsilon_n \xrightarrow{n\to\infty} 0$,

$$\dim(Q_{\epsilon_n}(\lambda_0)(X_{\epsilon_n})) > \dim(Q(\lambda_0)(X)).$$

From a previous exercise, for each $n \in \mathbb{N}$, there is a $u_n \in W(\lambda_0, A_{\epsilon_n})$ with $\|u_n\| = 1$ such that $\mathrm{dist}(u_n, E_{\epsilon_n} W(\lambda_0, A_0)) = 1$.

From the compact convergence we can suppose that $Q_{\epsilon_n}(\lambda_0)u_n = u_n \xrightarrow{E} Q_0(\lambda_0)u_0 = u_0$ and we have an absurdum since

$$1 \leqslant \|u_n - E_{\epsilon_n} Q_0(\lambda_0)u_0\|_{X_{\epsilon_n}} = \|Q_{\epsilon_n}(\lambda_0)u_n - E_{\epsilon_n} Q_0(\lambda_0)u_0\|_{X_{\epsilon_n}} \to 0.$$

So, we only need to prove that $Q_\epsilon(\lambda_0) \xrightarrow{CC} Q(\lambda_0)$ as $\epsilon \to 0$.

This follows from $Q_\epsilon(\lambda_0) \xrightarrow{EE} Q(\lambda_0)$, and from the compact convergence of $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$ as $\epsilon \to 0$, from the uniform boundedness of $\|(\zeta A_\epsilon^{-1} - I)^{-1}\|$ for $\zeta \in S_\delta(\lambda_0)$ and $\epsilon \in [0, \epsilon_0]$, given in the proof of a previous result, and from the formula

$$\begin{aligned} Q_\epsilon(\lambda_0) &= \frac{1}{2\pi i} \int_{|\zeta - \lambda_0| = \delta} (\zeta I - A_\epsilon)^{-1} d\zeta \\ &= A_\epsilon^{-1} \frac{1}{2\pi i} \int_{|\zeta - \lambda_0| = \delta} (\zeta A_\epsilon^{-1} - I)^{-1} d\zeta. \end{aligned}$$

(iv) Follows taking $u_\epsilon = Q_\epsilon(\lambda_0) E_\epsilon u$.
(v) Follows from the compact convergence of Q_ϵ to Q_0 proved in (iii). $\square$

Exercise

In the exercise Δ, show that the eigenvalues and eigenfunctions of A_ϵ converge to eigenvalues and eigenfunctions of A_0. Conclude that the convergence of eigenfunctions occur in the norm of $H^1(0,\pi)$.

Exercise (*)

In the exercise Δ, if λ_ϵ is an eigenvalue of A_ϵ, $0 \leq \epsilon \leq \epsilon_0$ and $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$, show that there is a $C > 0$ such that

$$|\lambda_\epsilon - \lambda_0| \leq C \|a_\epsilon - a_0\|_\infty^{\frac{1}{2}}.$$