SEMILINEAR FUNCTIONAL DIFFERENCE EQUATIONS WITH INFINITE DELAY

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ABSTRACT. We obtain boundness and asymptotic behavior of solutions for semilinear functional difference equations with infinite delay. Applications on Volterra difference equations with infinite delay are shown.

Keywords: functional difference equations; infinite delay; boundness; asymptotic behavior; Volterra difference equations.

CONTENTS

1. INTRODUCTION AND STATEMENT OF RESULTS

The theory of difference equations has grown at an accelerated pace in the last decades. It now occupies a key position in applicable analysis. Several aspects of the theory of functional difference equations can be understood as a proper generalization of the theory of ordinary difference equations. However, the fact that the state space for functional difference equations is infinite dimensional requires the development of methods and techniques coming from functional analysis (*e. g.* theory of semigroups of operators on Banach spaces, spectral theory, etc.) The idea

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of considering phase spaces for studying qualitative properties of functional difference equations was used first by Murakami [53] for the study of some spectral properties of the solution operator for linear Volterra difference systems and then by Elaydi *et al.* [35] for the study of asymptotic equivalence of bounded solutions of a homogeneous Volterra difference system and its perturbations.

Besides its theoretical interest, the study of abstract retarded functional difference equations in phase space has great importance in applications. For those reasons the theory of difference equations with infinite delay has drawn the attention of several authors. Properties of the solutions have been studied in several contexts. For example, invariant manifolds theory [50], convergence theory [17, 27, 28], discrete maximal regularity [30], asymptotic behavior [20,29,51], exponential dichotomy and robustness [13], stability [36,37], periodicity [8,25,32,42,54,59–61,63].

However, until now the literature concerning discrete maximal regularity for functional difference equations with infinite delay is too incipient and should be developed, so that to produce a significant progress in the theory of Volterra difference equations with infinite delay. We note that even in difference equations this new subject is limited essentially to a few articles. In the continuous case, it is well known that the study of maximal regularity is very useful for treating semilinear and quasilinear problems (*e. g.* see [6, 7, 14, 33] and the bibliography therein). Discrete maximal regularity has been studied by Blunck in [9, 10] for linear difference equations of first order; see also [40, 57, 58]. Kalton and Portal in [48] discussed maximal regularity of power-bounded operator and relate the discrete to the continuous time problem for analytic semigroups. Recently, three interesting articles were published, the first one by Geissert [38] concerning the maximal regularity for linear parabolic difference equations. The second one by Cuevas and Lizama [22], who obtained existence and stability of solutions for semilinear second-order difference equations on Banach spaces by using recent characterization, in terms of $\mathscr{R}$-boundness of discrete maximal regularity studied in [21]. The third one by Cuevas and Lizama [24] concerning the existence and stability of solutions for semilinear first-order difference equations via maximal regularity.

1.1. **Boundness.** We are concerned with the study of the existence of bounded solutions for the semilinear problem

$$x(n+1) = L(n, x_n) + f(n, x_\bullet), \quad n \geq 0, \tag{1.1}$$

by means of the knowledge of maximal regularity properties for the retarded linear functional equation (*e. g.* see Marakami's paper [52].)

$$x(n+1) = L(n, x_n), \quad n \geq 0, \tag{1.2}$$

where $L\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$ is a bounded linear map with respect to the second variable; $\mathscr{B}$ denotes an abstract phase space which we shall explain briefly later (see Hino *et al.* [47] for an outline of the general philosophy of such spaces); $x_{\bullet}$ denotes the $\mathscr{B}$-valued function defined by $n \mapsto x_n$, where x_n is the history function, which is defined by $x_n(\theta) = x(n+\theta)$ for all $\theta \in \mathbb{Z}^-$.

We assume the following condition:

Condition (A). $\{L(n,\cdot)\}$ is a uniformly bounded sequence of bounded linear operators mapping $\mathscr{B}$ into $\mathbb{C}^r$.That means that there is a constant $M > 1$ such that

$$|L(n,\varphi)| \leq M\|\varphi\|_{\mathscr{B}}, \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } \varphi \in \mathscr{B}.$$

Condition (A) plays a crucial role in the obtainment of a characterization of exponential dichotomy for retarded functional difference equations in the phase space $\mathscr{B}_\gamma$ ($\gamma > 0$) defined by

$$\mathscr{B}_\gamma \stackrel{\text{def}}{=} \left\{\varphi\colon \mathbb{Z}^- \to \mathbb{C}^r\colon \sup_{\theta\in\mathbb{Z}^-} \frac{|\varphi(\theta)|}{e^{-\gamma\theta}} < \infty\right\} \tag{1.3}$$

equipped with the norm

$$\|\varphi\|_{\mathscr{B}_\gamma} = \sup_{\theta\in\mathbb{Z}^-} \frac{|\varphi(\theta)|}{e^{-\gamma\theta}},$$

see [13, Th. 1.1].

The following result (Theorem 1.1) ensures the existence and uniqueness of bounded solutions which are in l^p under quite general hypotheses. Such a theory does not exist at this time. The framework for the proof of this theorem uses the exponential dichotomy of the solution operator of system (1.2) and a new approach based on discrete maximal regularity (see [30].)

To establish Theorem 1.1 we need to introduce the following condition:

Condition (B). The following assumptions hold

(B$_1$) The function $f(n,\cdot)\colon l^p(\mathbb{Z}^+;\mathscr{B}) \to \mathbb{C}^r$ satisfies a Lipschitz condition, that is, for all $\xi,\eta \in l^p(\mathbb{Z}^+;\mathscr{B})$ and $n \in \mathbb{Z}^+$ we get

$$|f(n,\xi) - f(\eta)| \leq \beta_f(n)\|\xi - \eta\|_p,$$

where $\beta_f \stackrel{\text{def}}{=} (\beta_f(n)) \in l^p(\mathbb{Z}^+)$;

(B$_2$) $f(\cdot,0) \in l^p(\mathbb{Z}^+;\mathbb{C}^r)$.

Theorem 1.1. *Assume that (A) is fulfilled and that the functions* $N(\cdot)$ *and* $M(\cdot)$ *given by Axiom (PS₁) are bounded. In addition assume that equation* (1.2) *has an exponential dichotomy on* $\mathscr{B}$ *with data* $(\alpha, K_{ed}, P(\cdot))$ *and Condition (B) holds. Suppose that the following condition holds*

$$2K_{ed}K_{\mathscr{B}} \sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\|\beta_f\|_p + e^{-\alpha} < 1, \tag{1.4}$$

where $K_{\mathscr{B}}$ *is the constant of Axiom (PS₂), see Section 2. Then for each* $\varphi \in P(0)\mathscr{B}$ *there is a unique bounded solution* y *of* (1.1) *with* $P(0)y_0 = \varphi$ *such that* $y_\bullet \in l^p(\mathbb{Z}^+;\mathscr{B})$, *in particular* $y \in l^p(\mathbb{Z}^+;\mathbb{C}^r)$. *Moreover, one has that the following a priori estimate for the solution*

$$\|y_\bullet\|_p \le C(\|\varphi\|_{\mathscr{B}} + \|f(\cdot,0)\|_p), \tag{1.5}$$

where $C > 0$, *and*

$$\|y_\bullet(\varphi) - y_\bullet(\psi)\|_p \le \frac{K_{ed}}{1 - e^{-a} - 2K_{ed}K_{\mathscr{B}}\sup_{m\ge 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\|\beta_f\|_p}\|\varphi-\psi\|_{\mathscr{B}}. \tag{1.6}$$

Estimate (1.6) *implies the continuity of the application* $\varphi \in P(0)\mathscr{B} \mapsto y_\bullet(\varphi) \in l^p(\mathbb{Z}^+;\mathscr{B})$.

Next, we are concerned with the initial value problem defined by the semilinear difference equation with infinite delay

$$x(n+1) = \mathscr{L}(x_n) + g(n, x_n), \quad n \ge 0, \tag{1.7}$$

with initial condition

$$x_0 = \varphi \in \mathscr{B}, \tag{1.8}$$

where $\mathscr{L}\colon \mathscr{B} \to \mathbb{C}^r$ is a bounded operator and $g\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$.

The following result ensures the existence and uniqueness of bounded solutions of problem (1.7)–(1.8). To study this initial value problem we assume the following condition:

Condition (C). The solution operator $T(n)$ of equation

$$x(n+1) = \mathscr{L}(x_n), \quad n \in \mathbb{Z}^+, \tag{1.9}$$

is uniformly bounded, that is, there is a constant K_T such that $\|T(n)\|_{\mathscr{B}\to\mathscr{B}} \le K_T$ for all $n \in \mathbb{Z}^+$.

Theorem 1.2. *Assume that (C) is fulfilled. Let* $g\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$ *be a function such that* $g(\cdot, 0)$ *is summable in* $\mathbb{Z}^+$ *and there exists a summable function* $l_g\colon \mathbb{Z}^+ \to \mathbb{R}^+$ *such that*

$$|g(n,\varphi) - g(n,\psi)| \le l_g(n)\|\varphi-\psi\|_{\mathscr{B}}, \tag{1.10}$$

for every $n \in \mathbb{Z}_+$ *and* $\varphi, \psi \in \mathscr{B}$. *Then there exists an unique bounded solution of the problem* (1.7)–(1.8).

A similar result was obtained by Henríquez [43] for functional differential equations.

1.2. **Weighted boundness and asymptotic behavior.** To state the next result we introduce the following condition.

Condition (D). Suppose that the following statements hold

(D_1) The function $g(n,\cdot)\colon \mathscr{B}\to\mathbb{C}^r$ satisfies a Lipschitz condition for all $n\in\mathbb{Z}^+$, that is for all $\varphi,\psi\in\mathscr{B}$ and $n\in\mathbb{Z}^+$, we have

$$|g(n,\varphi)-g(n,\psi)|\le l_g(n)\|\varphi-\psi\|_{\mathscr{B}},$$

where $l_g \stackrel{\text{def}}{=} (l_g(n))\in l_1$.

(D_2) $g(\cdot,0)\in l^1_{\alpha^\#}(\mathbb{Z}^+,\mathbb{C}^r)$ (see Section 2 for the definition of $l^1_{\alpha^\#}$).

We have the following result about weighted bounded solutions.

Theorem 1.3. *Assume that Condition (A) and (D) hold and that the functions $N(\cdot)$ and $M(\cdot)$ given by Axiom (PS$_1$) are bounded. Let $K^\#$ e $\alpha^\#$ be the constants of Theorem 2.5, $l^\infty_{\alpha^\#}$ $|||\cdot|||_{\alpha^\#}$ and $\|\cdot\|_{1,\alpha^\#}$ as in Definition 2.1. Then there is an unique weighted bounded solution y of the evolution equation*

$$y(n+1)=L(n,y_n)+g(n,y_n), \tag{1.11}$$

for $n\ge 0$ with $y_0=0$ and $y_\bullet\in l^\infty_{\alpha^\#}$ (see Definition 2.1). Moreover, we have the following a priori estimate for the solution

$$|||y_\bullet|||_{\alpha^\#}\le K^\# K_{\mathscr{B}} e^{-\alpha^\#}\|g(\cdot,0)\|_{1,\alpha^\#} e^{K^\# K_{\mathscr{B}} e^{-\alpha^\#}\|l_g\|_1}. \tag{1.12}$$

The exponential dichotomy gives us relevant information about the asymptotic relation between weighted bounded solutions of (1.2) and its perturbed system (1.11). Precisely we have the following theorem on asymptotics.

Theorem 1.4. *Assume that Condition (A) holds and that the functions $N(\cdot)$ and $M(\cdot)$ given by Axiom (PS$_1$) are bounded. Suppose also that equation* (1.2) *has an exponential dichotomy with data $(\alpha,K_{ed},P(\cdot))$ and that condition (D$_1$) holds with $g(\cdot,0)\in l^1_\alpha(\mathbb{Z}^+,\mathbb{C}^r)$. If*

$$K_{\mathscr{B}}K_{ed}e^{\alpha}\sup_{m\ge 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\|l_g\|_1<1, \tag{1.13}$$

then for any solution $z(n)$ of (1.2) *such that $z_\bullet\in l^\infty_\alpha$, there exists an unique $y(n)$ de* (1.11) *such that $y_\bullet\in l^\infty_\alpha$ and*

$$y(n)=z(n)+o(e^{\alpha n}),\quad n\to\infty. \tag{1.14}$$

The conversely is also true. Furthermore, the one-to-one correspondences $y_\bullet\mapsto z_\bullet$ e $z_\bullet\mapsto y_\bullet$ are continuous. ???(NO ORIGINAL, P 27, ESTAVA $y_\bullet\to x_\bullet$ ...)

1.3. **Asymptotic periodicity.** A very important aspect of the qualitative study of the solutions of difference equations is their periodicity and in general their asymptotic periodicity. Results in such direction cannot be deduced directly from the theory on the continuous case (*e. g.* [62].) Almost periodicity of a discrete function was first introduced by Walther [67, 68]. There is much interest in developing the periodicity study for difference equations. For details, including some applications and recent developments, see the monographs of Agarwal [1], Elaydi [34] and Agarwal and Wong [5], and the papers by Agarwal *et al.* [3, 4, 56], Corduneanu [16], Halanay [41] and Sugiyama [64]. Recently several works (see [59–61, 63]) have been devoted to studying the existence of almost periodic solutions of discrete systems with delay. The main method employed in these papers is to assume certain stability properties of a bounded solution.

On the other hand, we mention here the paper by del Campo *et al.* [32] where the authors established the existence of almost and asymptotic almost periodic solutions for functional difference equations with infinite delay of the form (1.7). The authors have assumed that the solution operator $T(\cdot)$ associated to the homogeneous linear system (1.9) is uniformly stable.

The class of functions $\xi\colon \mathbb{Z}^+ \to \mathscr{B}$ for which there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that $\lim_{n\to\infty}(\xi(n+\omega) - \xi(n)) = 0$ are called S-asymptotically ω-periodic solutions for retarded functional difference equations. The literature concerning S-asymptotically ω-periodic functions with values in Banach spaces is very new. This kind of functions have many applications in several problems, for example in the theory of functional differential equations, fractional differential equations, integral equations and partial differential equations. The concept of S-asymptotic ω-periodicity was introduced by Henríquez *et al.* [44, 45]. The article [45] concerns the development of a theory of this type of functions in a Banach space setting. In particular, the authors have established a relationship between S-asymptotically ω-periodic functions and the class of asymptotically ω-periodic functions. In [44] the authors study the existence and qualitative properties of S-asymptotically ω-periodic mild solutions for a class of abstract neutral functional differential equations with infinite delay. Related to the problem of existence of S-asymptotically ω-periodic solutions for ordinary differential equations described in finite dimensional spaces we cite [39, 49, 65, 69]. It is important to note the fact that a S-asymptotically ω-periodic function is not in general asymptotically ω-periodic (see [55]) and that the theory of almost periodic functions (see [15, 70]) have not been satisfactory to study this class of functions. Recently Cuevas and de Souza [18, 19] have treated the existence of S-asymptotically ω-periodic (mild) solutions for fractal integro-differential equations. ???? (Pg. 44

O RESPECTIVAMENTE NAO ESTA CLARO, VERIFICAR; AQUI EU REESCREVI SEM O RESPECTIVAMENTE) In [31], de Andrade and Cuevas studied the existence of S-asymptotically ω-periodic (mild) solutions to a first-order differential equation with linear part dominated by a Hille-Yosida operator with non-dense domain and Agarwal *et al.* [2] and Caicedo and Cuevas [11] studied the existence of such solutions to an abstract neutral integro-differential equation with unbounded delay. Cuevas and Lizama [23] have studied the existence of studied the existence of S-asymptotically ω-periodic solutions to a class of semilinear Volterra equations. In Hernández *et al.* [46] the authors studied the existence of S-asymptotically ω-periodic "classical" solutions for a class of abstract neutral functional differential equations with unbounded delay (see also [12, 26].)

The next result ensures the existence and uniqueness of a discrete S-asymptotically ω-periodic solution for the problem (1.7)–(1.8). To the knowledge of the authors until now there are no results in such direction for functional difference equations.

Theorem 1.5. *Assume that the solution operator of* (1.9) *is a strongly S-asymptotically ω-periodic semigroup (see Section 2.) Let g be a perturbation of* (1.9) *as in Theorem 1.2. Then there is a unique discrete S-asymptotically ω-periodic solution of the problem* (1.7)–(1.8) *for every $\varphi \in \mathscr{B}$.*

To establish our next result on the existence of discrete asymptotically ω-periodic solutions (see Definition 2.12) for the abstract problem (1.7)–(1.8) we introduce the another assumption.

Condition (E). The function $g\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$ satisfies a Lipschitz condition in $\mathscr{B}$, that is, there is a $K_g > 0$ such that

$$|g(n,\varphi) - g(n,\psi)| \le K_g \|\varphi - \psi\|_{\mathscr{B}}, \quad \varphi,\psi \in \mathscr{B}, \quad n \in \mathbb{Z}^+.$$

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Theorem 1.6. *Assume that the solution operator $T(n)$ of the Equation* (1.9) *decays exponentially, that is, there are constants $K^T \ge 1$ and $\alpha > 0$ such that*

$$\|T(n)\|_{\mathscr{B}\to\mathscr{B}} \le K^T e^{-\alpha n}, \quad n \in \mathbb{Z}^+. \tag{1.15}$$

Assume also that Condition (E) holds, $g(\cdot,0)$ is a bounded function and for every $B \subset \mathscr{B}$, $\lim_{n\to\infty} g(n,\varphi) - g(n+m,\varphi) = 0$ uniformly for $\varphi \in B$ and $m \in \mathbb{Z}^+$. If $K_{\mathscr{B}} K^T K_g + e^- \alpha < 1$ then there is a unique discrete asymptotically ω-periodic solution of the problem (1.7)–(1.8).

The paper is organized as follows. In Section 2 we explain the basic properties and we recall the definition of exponential dichotomy which is a natural tool in our setting. We also give some basic properties related with S-asymptotically ω-periodic functions. In Section 3 we prove theorems 1.1 and 1.2, while in Section 4 we prove theorems 1.3 and 1.4. Also we prove that under certain conditions there is a one-to-one correspondence between weighted bounded solutions of equation (1.2)[1] and its perturbation (1.11). In Section 5 we establish versions of theorems 1.1 and 1.3 which enable us to consider locally Lipschitz perturbations. In Section 6 we prove Theorem 1.5. ??? CORRIGIR AQUI PARA DIZER ONDE ESTA A PROVA DO TEO 1.6. ??? Finally in Section 7 we apply our results to Volterra difference equations. During the last few years Volterra equations have emerged vigorously. We observe that there is much interest in developing the qualitative theory for such equations. Our work is an interesting contribution to the Volterra equations theory.

2. Preliminaries and notations

We introduce certain notations which will be used along the paper without any further mentioning. To write out statements, we will often adopt the symbols and notation in a standard way as in [13, 20, 27–30, 35–37, 42, 50–53], for example. As usual, we denote by $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Z}^-$ the sets of all integers, all non-negative integers and non-positive integers, respectively. Let $\mathbb{C}^r$ be the r-dimensional complex Euclidean space with norm $|\cdot|$. We follow the terminology used in Murakami [52], thus the phase space $\mathscr{B} = \mathscr{B}(\mathbb{Z}^-, \mathbb{C}^r)$ is a Banach space with norm denoted by $\|\cdot\|_{\mathscr{B}}$ and it is assume to satisfy the following axioms.

Axiom (PS$_1$). There are a positive constant J and non-negative functions $N(\cdot)$ and $M(\cdot)$ on $\mathbb{Z}^+$ with the property that if $x\colon \mathbb{Z} \to \mathbb{C}^r$ is if a function such that $x_0 \in \mathscr{B}$, then for all $n \in \mathbb{Z}^+$

(i) $x_n \in \mathscr{B}$;
(ii) $J|x(n)| \le \|x_n\|_{\mathscr{B}} \le N(n) \sup_{0\le s\le n} |x(s)| + M(n)\|x_0\|_{\mathscr{B}}$.

Denote by $B(\mathbb{Z}^-, \mathbb{C}^r)$ the set of bounded functions from $\mathbb{Z}^-$ to $\mathbb{C}^r$.

Axiom (PS$_2$). The inclusion map $i\colon (B(\mathbb{Z}^-, \mathbb{C}^r), \|\cdot\|_\infty) \to (\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is continuous, that is, there is a constant $K_{\mathscr{B}} > 0$ such that $\|\varphi\|_{\mathscr{B}} \le K_{\mathscr{B}}\|\varphi\|_\infty$ for all $\varphi \in \mathscr{B}(\mathbb{Z}^-, \mathbb{C}^r)$.

From now on $\mathscr{B}$ will denote a phase space satisfying the axioms (PS$_1$) and (PS$_2$). Throughout this paper, for any number $1 \le p < \infty$ we shall consider the following spaces:

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Definition 2.1.

$$l^p(\mathbb{Z}^+,\mathscr{B}) \stackrel{\text{def}}{=} \{\xi\colon \mathbb{Z}^+ \to \mathscr{B} \,/\, \|\xi(n)\|_p^p \stackrel{\text{def}}{=} \textstyle\sum_{n=0}^{\infty} \|\xi(n)\|_{\mathscr{B}}^p < \infty\},$$

$$l^\infty(\mathbb{Z}^+,\mathscr{B}) \stackrel{\text{def}}{=} \{\xi\colon \mathbb{Z}^+ \to \mathscr{B} \,/\, \|\xi\|_\infty \stackrel{\text{def}}{=} \sup_{n\in\mathbb{Z}^+} \|\xi(n)\|_{\mathscr{B}} < \infty\},$$

$$l^\infty_\beta(\mathbb{Z}^+,\mathscr{B}) \stackrel{\text{def}}{=} \{\xi\colon \mathbb{Z}^+ \to \mathscr{B} \,/\, |||\xi|||_\beta \stackrel{\text{def}}{=} \sup_{n\in\mathbb{Z}^+} \|\xi(n)\|_{\mathscr{B}} e^{-\beta n} < \infty\},$$

$$l^1_\beta(\mathbb{Z}^+,\mathbb{C}^r) \stackrel{\text{def}}{=} \{\varphi\colon \mathbb{Z}^- \to \mathbb{C}^r \,/\, \|\varphi\|_{1,\beta} \stackrel{\text{def}}{=} \textstyle\sum_{n=0}^{\infty} |\varphi(n)| e^{-\beta n} < \infty\},$$

$$l^p(\mathbb{Z}^+,\mathbb{C}^r) \stackrel{\text{def}}{=} \{\varphi\colon \mathbb{Z}^+ \to \mathbb{C}^r \,/\, \|\varphi\|_p^p \stackrel{\text{def}}{=} \textstyle\sum_{n=0}^{\infty} |\varphi(n)|^p < \infty\}.$$

For any $n \geq \tau$ we define the operator $T(n,\tau)\colon \mathscr{B} \to \mathscr{B}$ by

$$T(n,\tau)\varphi = x_n(\tau,\varphi,0) \quad \varphi \in \mathscr{B}$$

where $x(\cdot,\tau,\varphi,0)$ denotes the solution of the homogeneous linear system (1.2) passing through (τ,φ). It is clear that the operator $T(n,\tau)$ is linear and by virtue of Axiom (PS_1) it is bounded on $\mathscr{B}$. We denote by $\|T(n,\tau)\|_{\mathscr{B}\to\mathscr{B}}$ the norm of the operator $T(n,\tau)$, which satisfies the semigroup properties

$$T(n,s)T(s,\tau) = T(n,\tau), \quad n \geq s \geq \tau, \qquad T(n,n) = I, \quad n \geq 0. \tag{2.1}$$

The operator $T(n,\tau)$ is called the solution operator of the homogeneous linear system (1.2) (see [52] for details.)

Definition 2.2 (Cardoso and Cuevas [13]). We say that Equation (1.2) (or his solution operator $T(n,\tau)$) has an *exponential dichotomy* on $\mathscr{B}$ with data $\alpha, K_{ed}, P(\cdot)$ if α, K_{ed} are positive numbers and $P(n)$, $n \in \mathbb{Z}^+$, are projections in $\mathscr{B}$ such that, letting $Q(n) - I - P(n)$:

(i) $T(n,\tau)P(\tau) = P(n)T(n,\tau), \quad n \geq \tau.$
(ii) The restriction $T(n,\tau)|\text{Range}(Q(\tau))$, $n \geq \tau$, is an isomorphism from $\text{Range}(Q(\tau))$ onto $\text{Range}(Q(n))$, and then we define $T(\tau,n)$ as its inverse mapping.
(iii) $\|T(n,\tau)\varphi\|_{\mathscr{B}} \leq K_{ed} e^{-\alpha(n-\tau)} \|\varphi\|_{\mathscr{B}}, \quad n \geq \tau, \quad \varphi \in P(\tau)\mathscr{B}.$
(iv) $\|T(n,\tau)\varphi\|_{\mathscr{B}} \leq K_{ed} e^{\alpha(n-\tau)} \|\varphi\|_{\mathscr{B}}, \quad \tau > n, \quad \varphi \in Q(\tau)\mathscr{B}.$

In what follows we consider the matrix function $E^0(t)$, $t \in \mathbb{Z}^-$, defined by

$$E^0(t) = \begin{cases} I \ (r \times r \text{ unit matrix}), & t = 0, \\ 0 \ (r \times r \text{ zero matrix}), & t < 0, \end{cases} \tag{2.2}$$

We denote by $\Gamma(n,s)$ the Green function associated with (1.2), that is,

$$\Gamma(t,s) = \begin{cases} T(n,s+1)P(s+1) & n-1 \geq s, \\ -T(n,s+1)Q(s+1) & s > n-1. \end{cases} \tag{2.3}$$

The following definition was introduced in [30].

Definition 2.3. We say that system (1.2) has *discrete maximal regularity* if for each $h \in l^p(\mathbb{Z}^+, \mathbb{C}^r)$ $(1 \le p < \infty)$ and each $\varphi \in P(0)\mathscr{B}$ the solution z of the boundary value problem

$$z(n+1) = L(n, z_n) + h(n), \quad n \ge 0, \tag{2.4}$$

$$P(0)z_0 = \varphi, \tag{2.5}$$

satisfies $z_\bullet \in l^p(\mathbb{Z}^+, \mathscr{B})$.

It was shown in [13] the following result:

Theorem 2.4. *Assume that system* (1.2) *has an exponential dichotomy on* $\mathscr{B}$ *with data* $(\alpha, K_{ed}, P(\cdot))$. *Then system* (1.2) *has discrete maximal regularity.*

Exponential boundness of the solution operator plays a key role in our results.

Theorem 2.5 (Exponential boundness of the solution operator)**.** *Assume that (A) is fulfilled. In addition suppose that the functions* $N(\cdot)$ *and* $M(\cdot)$ *given in Axiom (PS*$_1$*) are bounded*[2] *Then there are positive constants* $K^\#$ *and* $\alpha^\#$ *such that*

$$\|T(n,m)\|_{\mathscr{B}\to\mathscr{B}} \le K^\# e^{\alpha^\#(n-m)}, \quad n \ge m \ge 0. \tag{2.6}$$

A similar result was obtained by Cardoso and Cuevas [13] in the phase space $\mathscr{B}_\gamma$ given by (1.3)

Proof of Theorem 2.5. The proof is based on the argument of proof of [13, Prop. 2.1]. Without loss of generality we may assume that $M_L > \max\{1, 1/J\}$, where J is the constant given in Axiom (PS$_1$). Take now $N^I \overset{\text{def}}{=} \max\{\|N\|_\infty, \|M\|_\infty, 1\}$ and let $x(\cdot, m, \varphi)$ be the solution of the homogeneous system (1.2) passing through (m, φ). To prove (2.6), we observe that in view of Condition (A) and (PS$_1$), we get

$$\begin{aligned} \|T(m,n)\varphi\|_{\mathscr{B}} &\le N^\infty \left(\sum_{j=0}^{n-m} (M_L N^\infty)^j \right) \|\varphi\|_{\mathscr{B}} \\ &\le \frac{N^\infty}{M_L N^\infty - 1} (M_L N^\infty)^{n-m} \|\varphi\|_{\mathscr{B}}. \end{aligned}$$

This clearly implies (2.6) with

$$K^\# = N^\infty / (M_L N^\infty - 1) \quad \text{and} \quad \alpha^\# = \ln(M_L N^\infty). \qquad \square$$

[2] We note that conditions of this type have been previously considered in the literature. See for instance [29, 42, 66].

Proposition 2.6. *Under the conditions of Theorem 2.5, if system* (1.2) *has an exponential dichotomy with data* $(\alpha, K_{ed}, P(\cdot))$, *then*

(i) $\sup_{n\in\mathbb{Z}^+}\|P(n)\|_{\mathscr{B}\to\mathscr{B}} < \infty$.

(ii) $\mathrm{Range}(P(n)) = \{\varphi \in \mathscr{B} \colon e^{-\eta(n-m)}T(n,m)\varphi$ *is bounded for* $n \geq m\}$ *for any* $0<\eta<\alpha$.

(iii) *Let* $\hat{P}(0)$ *be a projection such that* $\mathrm{Range}(\hat{P}(0)) = \mathrm{Range}(P(0))$. *Then* (1.2) *has an exponential dichotomy on* $\mathbb{Z}^+$ *with data* $(\alpha, \hat{K}_{ed}, \hat{P}(\cdot))$, *where*

$$\hat{P}(n) = P(n) + T(n,0)\hat{P}(0)T(0,n)Q(n),$$
$$\hat{K}_{ed} = (K_{ed} + K_{ed}^2\|\hat{P}(0)\|_{\mathscr{B}\to\mathscr{B}})\sup_{m\geq 0}(1+\|P(n)\|_{\mathscr{B}\to\mathscr{B}}).$$

In addition, we have

(2.7)
$$\sup_{m\geq 0}\|\hat{P}(m)\|_{\mathscr{B}\to\mathscr{B}} \leq (1+K_{ed}^2\|\hat{P}(0)-P(0)\|_{\mathscr{B}\to\mathscr{B}})\sup_{m\geq 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}}).$$

Also one has

$$\hat{P}(n) = P(n) + o(1), \quad \text{as } n\to\infty. \tag{2.8}$$

Proof. (i) For a fixed $\tau > 0$, set

$$\gamma_\tau \stackrel{\text{def}}{=} \inf\{\|\varphi+\psi\|_{\mathscr{B}} \colon \varphi \in P(\tau)\mathscr{B},\ \psi\in Q(\tau)\mathscr{B},\ \|\varphi\|_{\mathscr{B}} = \|\psi\|_{\mathscr{B}} = 1\}.$$

If $\varphi\in\mathscr{B}$ is such that $P(\tau)\varphi\neq 0$ and $Q(\tau)\varphi \neq 0$, then

$$\gamma_\tau \leq \frac{1}{\|P(\tau)\varphi\|_{\mathscr{B}}}\left\|\varphi + \frac{\|P(n)\varphi\|_{\mathscr{B}} - \|Q(\tau)\varphi\|_{\mathscr{B}}}{\|Q(\tau)\varphi\|_{\mathscr{B}}}Q(\tau)\varphi\right\| \leq \frac{2\|\varphi\|_{\mathscr{B}}}{\|P(\tau)\varphi\|_{\mathscr{B}}}.$$

Hence,

$$\|P(\tau)\|_{\mathscr{B}\to\mathscr{B}} \leq \frac{2}{\gamma_\tau}.$$

It remains to show that there is a constant $C>0$ (independent of τ) such that $\gamma_\tau \geq C$. For this we consider $\varphi \in P(\tau)\mathscr{B}$, $\psi\in Q(\tau)\mathscr{B}$ with $\|\varphi\|_{\mathscr{B}} = \|\psi\|_{\mathscr{B}} = 1$. Taking advantage of the exponential boundness of the solution operator, we have

$$\|\varphi+\psi\|_{\mathscr{B}} \geq (K^\#)^{-1}e^{-\alpha^\#(n-\tau)}(K_{ed}^{-1}e^{\alpha(n-\tau)} - K_{ed}e^{-\alpha(n-\tau)}) \stackrel{\text{def}}{=} C_{n-\tau}$$

and hence $\gamma_\tau \geq C_{n-\tau}$. Obviously, $C_m > 0$ for m sufficiently large. Thus, $0 < C_m \leq \gamma_\tau$.

(ii) The inclusion

$$\mathrm{Range}(P(m)) \subset \{\varphi\in\mathscr{B}\colon e^{-n(n-m)}T(n,m)\varphi \text{ is bounded for } n\geq m\}$$

is obvious, while the conversely follows from

$$\|Q(m)\varphi\|_{\mathscr{B}} = \|T(m,n)Q(n)T(n,m)\varphi\|_{\mathscr{B}} \leq Ce^{(\alpha-\eta)(m-n)} \to 0 \quad \text{as } n\to\infty,$$

where C is a suitable constant.

(iii) It is easy to see that for $n \geq \tau$

$$T(n,\tau)\hat{P}(\tau) = \hat{P}(n)T(n,\tau).$$

We recall that the operator $T(n,\tau)$, $n \geq \tau$, is an isomorphism from $\hat{Q}(\tau)\mathscr{B}$ to $\hat{Q}(n)\mathscr{B}$. In fact, we define $T(\tau,n)$ as the inverse mapping which is given by

$$T(\tau,n)\hat{Q}(n)\varphi = T(\tau,n)Q(n)\varphi - T(\tau,0)\hat{P}(0)T(0,n)Q(n)\varphi.$$

On the other hand, if $n \geq \tau$ and $\varphi \in \hat{P}(\tau)\mathscr{B}$, then $T(n,\tau)\varphi$ is estimated by

$$\begin{aligned}\|T(n,\tau)\varphi\|_{\mathscr{B}} &\leq K_{ed}e^{-\alpha(n-\tau)}\|P(\tau)\|_{\mathscr{B}\to\mathscr{B}}\|\varphi\|_{\mathscr{B}} \\ &\quad + K_{ed}^2 e^{-\alpha(n-\tau)}\|\hat{P}(0)\|_{\mathscr{B}\to\mathscr{B}}\|Q(\tau)\|_{\mathscr{B}\to\mathscr{B}}\|\varphi\|_{\mathscr{B}} \\ &\leq \hat{K}_{ed}e^{-\alpha(n-\tau)}\|\varphi\|_{\mathscr{B}}.\end{aligned}$$

If $n < \tau$ e $\varphi \in \hat{Q}(\tau)\mathscr{B}$ and $\varphi \in \hat{Q}(\tau)\mathscr{B}$ then we estimate $T(n,\tau)\varphi$ by

$$\begin{aligned}\|T(n,\tau)\varphi\|_{\mathscr{B}} &= e^{\alpha(n-\tau)}(K_{ed} + K_{ed}^2\|\hat{P}(0)\|_{\mathscr{B}\to\mathscr{B}})\|Q(\tau)\|_{\mathscr{B}\to\mathscr{B}}\|\varphi\|_{\mathscr{B}} \\ &\leq \hat{K}_{ed}e^{\alpha(n-\eta)}\|\varphi\|_{\mathscr{B}}.\end{aligned}$$

From

$$\|\hat{P}(n) - P(n)\|_{\mathscr{B}\to\mathscr{B}} \leq K_{ed}^2 e^{-2\alpha n}\sup_{m\geq 0}(1 + \|P(m)\|_{\mathscr{B}\to\mathscr{B}})\|\hat{P}(0) - P(0)\|_{\mathscr{B}\to\mathscr{B}}$$

it is easy to see that (2.7) and (2.8) hold. This completes the proof of Proposition 2.6. □

Let X and Y be two Banach spaces. The notation $\mathscr{L}(X,Y)$ stands for the space of bounded linear operators from X into Y endowed with the uniform operator topology, and we abbreviate to $\mathscr{L}(X)$ is $X = Y$.

Theorem 1.5 ensures the existence and uniqueness of a discrete S-asymptotically ω-periodic solution of (1.7)–(1.8). We recall the definition of S-asymptotic ω-periodicity.

Definition 2.7 (Henríquez *et al.* [45])**.** A sequence $\xi \in l^\infty(\mathbb{Z}^+,\mathscr{B})$ is called *(discrete) S-asymptotically ω-periodic* if there is $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that $\lim_{n\to\infty}(\xi(n+\omega) - \xi(n)) = 0$. In this case we say that ω is an asymptotic period of ξ. ??? REMOVI A REPETICAO NO FINAL ???

In this work the notation $\mathrm{SAP}_\omega(\mathscr{B})$ stands for the subspace of $l^\infty(\mathbb{Z}^+,\mathscr{B})$ consisting of all the (discrete) S-asymptotically ω-periodic sequences. From [45, Prop. 3.5], $\mathrm{SAP}_\omega(\mathscr{B})$ is a Banach space.

Definition 2.8 (Henríquez *et al.* [45])**.** A strongly continuous function $F\colon \mathbb{Z}^+ \to \mathscr{L}(\mathscr{B})$ is said to be *strongly S-asymptotically periodic* if for each $\varphi \in \mathscr{B}$, there is $\omega_\varphi \in \mathbb{Z}^+ \setminus \{0\}$ such that $F(\cdot)\varphi$ is S-asymptotically ω_φ-periodic.

The function F is said *strongly S-asymptotically ω-periodic* if $F(\cdot)\varphi$ is S-asymptotically ω-periodic for all $\varphi \in \mathscr{B}$. ??? ALTEREI A DEFINICAO PARA ALGO QUE ME PARECEU O CORRETO ???

Definition 2.9 (Henríquez *et al.* [45])**.** A continuous function $g\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$ is said *uniformly S-asymptotically ω-periodic on bounded sets* if for every bounded subset B of $\mathscr{B}$, the set $\{g(n,\varphi)\colon n \in \mathbb{Z}^+, \varphi \in B\}$ is bounded and $\lim_{n\to\infty}(g(n,\varphi) - g(n+\omega,\varphi)) = 0$ uniformly on $\varphi \in B$.

Definition 2.10 (Henríquez *et al.* [45])**.** A function $g\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$ is said *uniformly asymptotically continuous on bounded sets* if for every $\epsilon > 0$ and every bounded set $B \subset \mathscr{B}$, there are $K_{\epsilon,B} \geq 0$ and $\delta_{\epsilon,B} \geq 0$ such that $|g(n,\varphi) - g(n,\psi)| < \epsilon$ for all $n \geq K_{\epsilon,B}$ and all $\varphi, \psi \in B$ with $\|\varphi - \psi\| < \delta_{\epsilon,B}$.

Lemma 2.11 (Henríquez *et al.* [45])**.** *Let $g\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$ be uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $\xi\colon \mathbb{Z}^+ \to \mathscr{B}$ be a discrete S-asymptotically ω-periodic function. Then the function $g(\cdot,\xi(\cdot))$ is discrete S-asymptotically ω-periodic.*

Definition 2.12 (Henríquez *et al.* [45])**.** A sequence $\xi \in l^\infty(\mathbb{Z}^+,\mathscr{B})$ is called discrete asymptotically ω-periodic if there exists a ω-periodic function $\eta \in l^\infty(\mathbb{Z}^+,\mathscr{B})$ and $\chi \in C_0(\mathbb{Z}^+,\mathscr{B})$ such that $\xi = \eta + \chi$. (Here $C_0(\mathbb{Z}^+,\mathscr{B})$ denotes the subspace of $l^\infty(\mathbb{Z}^+,\mathscr{B})$ of those ξ such that $\lim_{n\to\infty} \|\xi(n)\|_{\mathscr{B}} = 0$.)

Remark 2.13. Note that it is possible to exhibit a function that is S-asymptotically ω-periodic for every $\omega \in \mathbb{Z}^+ \setminus \{0\}$ but not asymptotically periodic. See [45, 55].

3. Boundness

In this section we will prove theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let ξ be a sequence in $l^p(\mathbb{Z}^+,\mathscr{B})$. Using Condition (B) we obtain that the function $g(\cdot) = f(\cdot,\xi)$ is in $l^p(\mathbb{Z}^+,\mathbb{C}^r)$. In fact we have

$$\begin{aligned}\|g\|_p^p &= \sum_{n=0}^{\infty} |f(n,\xi)|^p \\ &\leq \sum_{n=0}^{\infty} (|f(n,\xi) - f(n,0)| + |f(n,0)|)^p \\ &\leq 2^p \sum_{n=0}^{\infty} |f(n,\xi) - f(n,0)|^p + 2^p \sum_{n=0}^{\infty} |f(n,0)|^p \\ &\leq 2^p \sum_{n=0}^{\infty} \beta_f(n)^p \|\xi\|_p^p + 2^p \|f(\cdot,0)\|_p^p.\end{aligned}$$

Hence

$$\|g\|_p \le 2(\|\beta_f\|_p\|\xi\|_p + \|f(\cdot,0)\|_p),$$

proving that $g \in l^p(\mathbb{Z}^+,\mathscr{B})$.

If $\varphi \in P(0)\mathscr{B}$, by Theorem 2.4, system (1.2) has discrete maximal regularity, so the Cauchy problem

$$\begin{cases} z(n+1) = L(n,z_n) + g(n), & n \in \mathbb{Z}^+, \\ P(0)z_0 = \varphi, \end{cases} \tag{3.1}$$

has an unique solution z such that $z_\bullet \in l^p(\mathbb{Z}^+,\mathscr{B})$ which is given by

$$z_n = [\mathscr{K}\xi](n) = T(n,0)P(0)\varphi + \sum_{s=0}^{\infty} \Gamma(n,s)E^0(f(s,\xi)).$$

We now show that the operator $\mathscr{K}: l^p(\mathbb{Z}^+,\mathscr{B}) \to l^p(\mathbb{Z}^+,\mathscr{B})$ has a unique fixed point. Let ξ and η be in $l^p(\mathbb{Z}^+,\mathscr{B})$. In view of Condition (B) we have that

$$\begin{aligned}
\|\mathscr{K}\xi - \mathscr{K}\eta\|_p &= \left[\sum_{n=0}^{\infty}\left\|\sum_{s=0}^{\infty}\Gamma(n,s)E^0(f(s,\xi)-f(s,\eta))\right\|_{\mathscr{B}}^p\right]^{1/p} \\
&\le K_{ed}K_{\mathscr{B}} \sup_{m\in\mathbb{Z}^+}(1+\|P(n)\|_{\mathscr{B}\to\mathscr{B}}) \\
&\quad \times \left[\sum_{n=0}^{\infty}\left(\sum_{s=0}^{\infty}e^{-\alpha|n-(s+1)|}\beta_f(s)\right)^p\right]^{1/p}\|\xi-\eta\|_p \\
&\le K_{ed}K_{\mathscr{B}} \sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}}) \\
&\quad \times \left[\sum_{n=0}^{\infty}\left(\frac{2}{1-e^{-\alpha}}\right)^{p/q}\left(\sum_{s=0}^{\infty}e^{-\alpha|n-(s+1)|}\beta_f(s)^p\right)\right]^{1/p}\|\xi-\eta\|_p \\
&\le K_{ed}K_{\mathscr{B}} \sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\left(\frac{2}{1-e^{-\alpha}}\right)^{1/q} \\
&\quad \times \left[\sum_{s=0}^{\infty}\left(\frac{2}{1-e^{-\alpha}}\right)\beta_f(s)^p\right]^{1/p}\|\xi-\eta\|_p \\
&\le 2K_{ed}K_{\mathscr{B}} \sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})(1-e^{-\alpha})^{-1}\|\beta_f\|_p\|\xi-\eta\|_p. \qquad (3.2)
\end{aligned}$$

By (1.4) and the contraction principle, it follows that $\mathscr{K}$ has a unique fixed point $\xi \in l^p(\mathbb{Z}^+,\mathscr{B})$. The uniqueness of solutions is reduced to the uniqueness of the fixed point of map $\mathscr{K}$. Indeed, let $y = y(n,0,\varphi)$ be a solution of Equation (1.1) with $P(0)y_0 = \varphi$. Considering $\xi(n) = [\mathscr{K}y_\bullet](n)$, it follows from a straight forward computation that

$$\xi(n) = T(n,0)\xi(0) + \sum_{s=0}^{n-1} T(n,s+1)E^0(f(s,y_\bullet)), \quad n \ge 0.$$

Define

$$x(n) = \begin{cases} [\xi(n)](0), & n \geq 0, \\ [\xi(0)](n), & n < 0. \end{cases}$$

Applying [29, Lemma 2.8], $x(n)$ solves the evolution equation

$$\begin{cases} x(n+1) = L(n, x_\bullet) + f(n, y_\bullet), & n \geq 0, \\ P(0)x_0 = \varphi, \end{cases}$$

together with the relation $x_n = \xi(n)$, $n \geq 0$. Put $z_n = x_n - y_n$, so $z(n)$ is solution of (1.2) for $n \geq 0$, with $P(0)z_0 = 0$. Using Theorem 2.1 in [52], we get $z_n = T(n,0)z_0, \quad n \geq 0$.

Now, by property (ii) of Definition 2.2, we obtain that

$$z_0 = T(0,n)Q(n)z_n, \quad n \geq 0.$$

By property (iv) of Definition 2.2 and taking into account Proposition 2.6 (i), we obtain

$$\|z_0\|_{\mathscr{B}} \leq K_{ed} e^{-\alpha n} \sup_{m \geq 0} (1 + \|P(m)\|_{\mathscr{B} \to \mathscr{B}}) \|z_\bullet\|_\infty, \quad n \geq 0.$$

We conclude that $z_0 = 0$ and hence $z_n = 0$, what implies the uniqueness of solutions of ??? (1.1) EXPECIFIQUEI???.

Let ξ be the unique fixed point of $\mathscr{K}$. From Condition (B) and (3.2), we have

$$\begin{aligned}
\|\xi\|_p &= \left[\sum_{n=0}^{\infty} \left\| T(n,0)P(0)\varphi + \sum_{s=0}^{\infty} \Gamma(n,s) E^0(f(s,\xi)) \right\|_{\mathscr{B}}^p \right]^{1/p} \\
&\leq \left[\sum_{n=0}^{\infty} \| T(n,0)P(0)\varphi \|_{\mathscr{B}}^p \right]^{1/p} \\
&\quad + \left[\sum_{n=0}^{\infty} \left\| \sum_{s=0}^{\infty} \Gamma(n,s) E^0(f(s,\xi)) \right\|_{\mathscr{B}}^p \right]^{1/p} \\
&\leq K_{ed} \left[\sum_{j=0}^{\infty} e^{-\alpha p j} \right]^{1/p} \|\varphi\|_{\mathscr{B}} \\
&\quad + 2K_{ed} K_{\mathscr{B}} \sup_{m \geq 0} (1 + \|P(m)\|_{\mathscr{B} \to \mathscr{B}})(1 - e^{-\alpha})^{-1} \\
&\quad \times \left[\sum_{s=0}^{\infty} |f(s,\xi)|^p \right]^{1/p} \\
&\leq K_{ed}(1 - e^{-\alpha})^{-1} \|\varphi\|_{\mathscr{B}} \\
&\quad + 2K_{ed} K_{\mathscr{B}} \sup_{m \geq 0} (1 + \|P(m)\|_{\mathscr{B} \to \mathscr{B}})(1 - e^{-\alpha})^{-1} \\
&\quad \times (\|\beta_f\|_p \|\xi\|_p + \|f(\cdot,0)\|_p),
\end{aligned}$$

whence

$$\|\xi\|_p \le \frac{K_{ed}\max\{1,2K_{\mathscr{B}}\}\sup_{m\ge 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})}{1-e^{-\alpha}-2K_{ed}K_{\mathscr{B}}\sup_{m\ge 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\|\beta_f\|_p} \times \Big[\|\varphi\|_{\mathscr{B}}+\|f(\cdot,0)\|_p\Big].$$

Our next step is to show (1.6), and we argue as follows.

$$\begin{aligned}
&\|y_\bullet(\varphi)-y_\bullet(\psi)\|_p \\
&\quad\le K_{ed}(1-e^{-\alpha})^{-1}\|\varphi-\psi\|_{\mathscr{B}} \\
&\qquad+2K_{ed}K_{\mathscr{B}}\sup_{m\ge 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})(1-e^{-\alpha})^{-1} \\
&\qquad\times\left[\sum_{s=0}^{\infty}\big|f(s,y_\bullet(\varphi))-f(s,y_\bullet(\psi))\big|^p\right]^{1/p} \\
&\quad\le K_{ed}(1-e^{-\alpha})^{-1}\|\varphi-\psi\|_{\mathscr{B}} \\
&\qquad+2K_{ed}K_{\mathscr{B}}\sup_{m\ge 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})(1-\alpha^{-\alpha})^{-1}\|\beta_f\|\,\|y_\bullet(\varphi)-y_\bullet(\psi)\|_p.
\end{aligned}$$

It is easy to see that the desired bounds, (1.6), follow from the above inequality. This ends the proof of the theorem. □

Example 3.1. Assume that (A) is fulfilled for the phase space $\mathscr{B}_\gamma$ (see (1.3)) and suppose that (1.2) has an exponential dichotomy on $\mathscr{B}_\gamma$. Let $\{\tilde{\mathscr{L}}(n,\cdot)\}_{n\in\mathbb{Z}^+}$ be a sequence of bounded linear operators from $\mathscr{B}_\gamma$ into $\mathbb{C}^r$. If $\sup_{n\in\mathbb{Z}^+}\|\tilde{\mathscr{L}}(n,\cdot)\|_{\mathscr{B}_\gamma\to\mathbb{C}^r}$ is sufficiently small, then by the robustness of the exponential dichotomy (see [13, Theorem 1.3]) we get that equation

$$x(n+1)=L(n,x_n)+\tilde{L}(n,x_n),\quad n\ge 0,$$

has an exponential dichotomy (on $\mathscr{B}_\gamma$) as well for suitable data $(\alpha_*,K_*,P_*(n))$.

Next, assume that Condition (B) holds and that

$$2K_*\sup_{m\in\mathbb{Z}^+}(1+\|P_*(m)\|_{\mathscr{B}_\gamma\to\mathscr{B}_\gamma})\|\beta_f\|_p+e^{-\alpha_*}<1.$$

Then by Theorem 1.1 for each $\varphi\in P_*(0)\mathscr{B}_\gamma$ there is an unique bounded solution x of the initial value problem

$$\begin{cases} x(n+1)=L(n,x_n)+\tilde{\mathscr{L}}(n,x_n)+f(n,x_\bullet), & n\in\mathbb{Z}^+,\\ P_*(0)x_0=\varphi, \end{cases}$$

such that $x_\bullet\in l^p(\mathbb{Z}^+,\mathscr{B}_\gamma)$, and in particular $x\in l^p(\mathbb{Z}^+,\mathbb{C}^r)$. Moreover, we have the following *a priori* estimate for the solution:

$$\|x_\bullet\|_p\le C(\|\varphi\|_{\mathscr{B}_\gamma}+\|f(\cdot,0)\|_p),$$

where $C>0$.

Proof of Theorem 1.2. We introduce the space $l^\infty(\mathbb{Z}^+,\mathbb{R})$ endowed with the norm of uniform convergence. We consider $l^\infty(\mathbb{Z}^+,\mathbb{R})$ provided with the pointwise order relation $\preceq$ and define the map $m\colon l^\infty(\mathbb{Z}^+,\mathscr{B}) \to l^\infty(\mathbb{Z}^+,\mathbb{R})$ by

$$m(\xi)(n) \stackrel{\text{def}}{=} \sup_{0\le s\le n} \|\xi(s)\|_{\mathscr{B}}, \quad \xi \in l^\infty(\mathbb{Z}^+,\mathscr{B}), \quad n \in \mathbb{Z}^+.$$

It is easy to see from this construction that the following properties hold:

(i) For all $\xi \in l^\infty(\mathbb{Z}^+,\mathscr{B})$, $0 \preceq m(\xi)$.

(ii) The norm in $l^\infty(\mathbb{Z}^+,\mathbb{R})$ is monotonic with respect to the order $\preceq$, that is, if

$$0 \preceq f_1 \preceq f_2 \quad \Longrightarrow \quad \|f_1\|_\infty \le \|f_2\|_\infty.$$

(iii) $\|m(\xi)\|_\infty = \|\xi\|_\infty$, for all $\xi \in l^\infty(\mathbb{Z}^+,\mathscr{B})$.

Now, let us consider $\varphi \in \mathscr{B}$ fixed and let us introduce the de Banach

$$l_0^\infty = \{\xi \in l^\infty(\mathbb{Z}^+,\mathscr{B}) : \xi(0) = 0\}$$

endowed with the norm of uniform convergence. We define on l_φ^∞ an operator $\mathscr{A}$ by

$$[\mathscr{A}\xi](n) = \sum_{s=0}^{n-1} T(n-s-1)E^0(g(s,\xi(s)) + T(s)\varphi). \tag{3.3}$$

We will show initially that $\mathscr{A}\xi \in l_\varphi^\infty$. In fact, we have the following estimates.

$$\begin{aligned}\|[\mathscr{A}\xi](n)\|_{\mathscr{B}} &\le K_{\mathscr{B}} \sum_{s=0}^{n-1} \|T(n-s-1)\|_{\mathscr{B}\to\mathscr{B}} |g(s,\xi(s) + T(s)\varphi)| \\ &\le K_T K_{\mathscr{B}} \sum_{s=0}^{n-1} l_g(s)\|\xi(s) + T(s)\varphi\|_{\mathscr{B}} + K_T K_{\mathscr{B}} \sum_{s=0}^{n-1} |g(s,0)|.\end{aligned}$$

Hence we obtain

$$\|\mathscr{A}\xi\|_\infty \le K_T K_{\mathscr{B}} \|l_g\|_1 [\|\xi\|_\infty + ???K_T\|\varphi\|] + K_T K_{\mathscr{B}} \|g(\cdot,0)\|_1.$$

Furthermore, for $\xi,\eta \in l_0^\infty$ the inequality

$$\|[\mathscr{A}\xi](n) - [\mathscr{A}\eta](n)\|_{\mathscr{B}} \le K_T K_{\mathscr{B}} \sum_{s=0}^{n-1} l_g(s)\|\xi(s) - \eta(s)\|_{\mathscr{B}}, \tag{3.4}$$

shows that $\mathscr{A} : l_0^\infty \to l_0^\infty$ is a continuous map.

On the other hand, we define the linear map $\mathscr{C} : l^\infty(\mathbb{Z}^+,\mathbb{R}) \to l^\infty(\mathbb{Z}^+,\mathbb{R})$ by the expression

$$[\mathscr{C}\alpha](n) = K_T K_{\mathscr{B}} \sum_{s=0}^{n-1} l_g(s)\alpha(s), \quad n \in \mathbb{Z}^+. \tag{3.5}$$

It is easy to show that $\mathscr{C}$ is a bounded linear operator on $l^\infty(\mathbb{Z}^+,\mathbb{R})$. In fact, we can infer that

$$\|\mathscr{C}\|_{l^\infty\to l^\infty} \le K_T K_{\mathscr{B}} \|l_g\|_1.$$

Moreover, $\mathscr{C}$ is a completely continuous map. To establish this assertion, for each $\epsilon > 0$ and $n_1 \in \mathbb{N}$ sufficiently large such that $K_T K_{\mathscr{B}} \sum_{s=n_1}^{\infty} l_g(s) \le \epsilon$ and, for each $\alpha \in l^\infty(\mathbb{Z}^+,\mathbb{R})$ with $\|\alpha\|_\infty \le 1$, we define the functions

$$[\mathscr{C}_1(\alpha)](n) = \begin{cases} K_T K_{\mathscr{B}} \sum_{s=0}^{n-1} l_g(s)\alpha(s), & 0 \le n \le n_1, \\ K_T K_{\mathscr{B}} \sum_{s=0}^{n_1-1} l_g(s)\alpha(s), & n \ge n_1, \end{cases}$$

and

$$[\mathscr{C}_2(\alpha)](n) = \begin{cases} 0, & 0 \le n \le n_1, \\ K_T K_{\mathscr{B}} \sum_{s=n_1}^{n-1} l_g(s)\alpha(s), & n \ge n_1. \end{cases}$$

We can infer that $K_\epsilon \overset{\text{def}}{=} \{\mathscr{C}_1(\alpha) \colon \|\alpha\|_\infty \le 1\}$ is relatively compact in $l^\infty(\mathbb{Z}^+,\mathbb{R})$. Since

$$[\mathscr{C}(\alpha)](n) = [\mathscr{C}_1(\alpha)](n) + [\mathscr{C}_2(\alpha)](n), \quad n \in \mathbb{Z}^+,$$

we can affirm that

$$\{\mathscr{C}(\alpha) \colon \|\alpha\|_\infty \le 1\} \subseteq K_\epsilon + \{\beta \colon \beta \in l^\infty(\mathbb{Z}^+,\mathbb{R}),\ \|\beta\|_\infty \le \epsilon\},$$

what shows that the set $\{\mathscr{C}(\alpha) \colon \|\alpha\|_\infty \le 1\}$ is relatively compact in $l^\infty(\mathbb{Z}^+,\mathbb{R})$ and, in turn, that $\mathscr{C}$ is completely continuous.

Moreover, since the point spectrum $\sigma_p(\mathscr{C}) = \{0\}$, the spectral radius of $\mathscr{C}$ is equal to zero. On the other hand, the operator $\mathscr{C}$ is increasing with respect to the order $\preceq$, that is, if $0 \preceq \alpha_1 \preceq \alpha_2$ then $\mathscr{C}\alpha_1 \preceq \mathscr{C}\alpha_2$.

Now, taking into account (3.4), we observe that

$$\|(\mathscr{A}\xi - \mathscr{A}\eta)(n)\|_{\mathscr{B}} \le K_T K_{\mathscr{B}} \sum_{s=0}^{n-1} l_g(s) \sup_{0\le j\le s} \|\xi(j) - \eta(j)\|_{\mathscr{B}},$$

so that

$$\begin{aligned} m(\mathscr{A}\xi - \mathscr{A}\eta)(n) &= \sup_{0\le k\le n} \|(\mathscr{A}\xi - \mathscr{A}\eta)(n)\|_{\mathscr{B}}, \\ &\le K_T K_{\mathscr{B}} \sup_{0\le k\le n} \sum_{s=0}^{k-1} l_g(s) m(\xi - \eta)(s) \\ &\le K_T K_{\mathscr{B}} \sum_{s=0}^{n-1} l_g(s) m(\xi - \eta)(s) \\ &= [\mathscr{C} m(\xi - \eta)](n), \end{aligned}$$

for each $n \in \mathbb{Z}^+$. Therefore, for all $\xi, \eta \in l^\infty_\varphi$,

$$m(\mathscr{A}\xi - \mathscr{A}\eta) \preceq \mathscr{C} m(\xi - \eta).$$

Thus the maps $\mathscr{A}, \mathbb{C}$ e m satisfy all the conditions of [43, Theorem 1], which implies that $\mathscr{A}$ has a unique fixed point. Finally, we observe that the uniqueness of the solutions ??? ESPECIICANDO ??? of the problem (1.7)–(1.8) ??? follows from the uniqueness of the fixed point of the map $\mathscr{A}$. This completes the proof of Theorem 1.2. □

Corollary 3.2. *Assume that the solution operator of* (1.9), $T(n)$, *decays exponentially (see* (1.15) *and Section 7.) Let g be a perturbation of* (1.9) *as in Theorem 1.2, and that one of the following conditions holds:*

(i) $g(\cdot,0)e^{\alpha\cdot} \in l^1(\mathbb{Z}^+,\mathbb{C}^r)$ *and* $l_g(\cdot)e^{\alpha\cdot} \in l^1(\mathbb{Z}^+,\mathbb{R}^+)$.

(ii) $g(\cdot,0) \in l^\infty(\mathbb{Z}^+,\mathbb{C}^r)$ *and* $l_g(n) \equiv l_g$ $\forall n \in \mathbb{Z}^+$, *where* l_g *is a positive constant such that* $K_{\mathscr{B}}K^T l_g(1-e^{-\alpha})^{-1} < 1$, *with* $K_{\mathscr{B}}$ *being that of Axiom* (PS_2) *and* K^T *ans* α *given by* (1.5).

Then there exists an unique bounded solution for the problem (1.7)–(1.8).

Remark 3.3. ??? TOTALMENTE CONFUSO ???

We observe that in the case of above condition (i) in Corollary 3.2, using [13, Cor. 6.2], we can get a similar result ??? CONFUSING ???, but we need to impose a strong condition on the function $l_g(\cdot)e^{\alpha\cdot}$; namely that such function has a l^1-norm sufficiently small. On the other hand, Corollary 6.2 in [13] cannot assure uniqueness of solutions to (1.7)–(1.8). Therefore our above result generalizes substantially [13, Cor. 6.2].

4. Weighted boundness and asymptotic behavior

In this section we shall prove theorems 1.3 and 1.4.

Proof of Theorem 1.3 (page 5). We define the operator Ω on $l^\infty_{\alpha^\#}$ by

$$[\Omega\xi](n) = \sum_{s=0}^{n-1} T(n,s+1)E^0(g(s,\xi(s))), \quad \xi \in l^\infty_{\alpha^\#}. \tag{4.1}$$

We now show that the operator $\Omega\colon l^\infty_{\alpha^\#} \to l^\infty_{\alpha^\#}$ has an unique fixed point. We observe that Ω is well defined. In fact, we use Condition (D) and Theorem 2.5 to obtain

$$\begin{aligned}\left\|[\Omega\xi](n)\right\|_{\mathscr{B}} e^{-\alpha^\# n} &\le K^\# K_{\mathscr{B}} e^{-\alpha^\#} \sum_{s=0}^{n-1} \left|g(s,\xi(s))\right| e^{-\alpha^\# s} \\ &\le K^\# K_{\mathscr{B}} e^{-\alpha^\#} \left[\sum_{s=0}^{n-1} l_g(s)\left\|\xi(s)\right\|_{\mathscr{B}} e^{-\alpha^\# s} + \sum_{s=0}^{n-1}\left|g(s,0)\right| e^{-\alpha^\# s}\right]\end{aligned}$$

whence

$$\left|\!\left|\!\left|[\Omega\xi](n)\right|\!\right|\!\right|_{\alpha^\#} \le K^\# K_{\mathscr{B}} e^{-\alpha^\#}\left[\|l_g\|_1 |||\xi(s)|||_{\alpha^\#} + \|g(\cdot,0)\|_{1,\alpha^\#}\right]. \tag{4.2}$$

It proves that the space $l^\infty_{\alpha^\#}$ is invariant under Ω.

Let ξ and η be in $l^\infty_{\alpha^\#}$. In view of Theorem 2.5 and Condition (D_1), we have initially

$$\|[\Omega\xi](n)-[\Omega\eta](n)\|_{\mathscr{B}}e^{-\alpha^\# n} \le K^\# K_{\mathscr{B}} e^{-\alpha^\#}\left(\sum_{s=0}^{n-1} l_g(s)\right)|||\xi-\eta|||_{\alpha^\#}$$

$$\le K^\# K_{\mathscr{B}} e^{-\alpha^\#}\|l_g\|_1 |||\xi-\eta|||_{\alpha^\#}. \tag{4.3}$$

Next we consider the iterates of the operator Ω. We observe from (4.3) that

$$\begin{aligned}
&\|[\Omega^2\xi](n)-[\Omega^2\eta](n)\|_{\mathscr{B}}e^{-\alpha^\# n}\\
&\qquad\le K^\# K_{\mathscr{B}} e^{-\alpha^\#}\sum_{s=0}^{n-1} l_g(s)\|[\Omega\xi](n)-\Omega\eta](n)\|_{\mathscr{B}}e^{-\alpha^\# n}\\
&\qquad\le [K^\# K_{\mathscr{B}} e^{-\alpha^\#}]^2\left(\sum_{s=0}^{n-1} l_g(s)\left(\sum_{i=0}^{s-1} l_g(i)\right)\right)|||\xi-\eta|||_{\alpha^\#}\\
&\qquad\le \frac{1}{2}[K^\# K_{\mathscr{B}} e^{-\alpha^\#}]^2\left(\sum_{s=0}^{n-1} l_g(s)\right)^2 |||\xi-\eta|||_{\alpha^\#}.
\end{aligned}$$

Therefore,

$$|||[\Omega^2\xi](n)-[\Omega^2\eta](n)|||_{\alpha^\#} \le \frac{1}{2}[K^\# K_{\mathscr{B}} e^{-\alpha^\#}\|l_g\|_1]^2|||\xi-\eta|||_{\alpha^\#}.$$

In general, proceeding by induction, we can assert that

$$|||[\Omega^n\xi](n)-[\Omega^n\eta](n)|||_{\alpha^\#} \le \frac{1}{n!}[K^\# K_{\mathscr{B}} e^{-\alpha^\#}\|l_g\|_1]^n|||\xi-\eta|||_{\alpha^\#}.$$

The above estimates imply that the operator Ω^n is a contraction for n sufficiently large. Therefore, Ω has an unique fixed point in $l^\infty_{\alpha^\#}$. The uniqueness of solutions ??? of (1.11) ??? is reduced to the uniqueness of the fixed point of the map Ω.

Let ξ be the unique fixed point of Ω. Then by Condition (D) we have

$$\|\xi(n)\|_{\mathscr{B}}e^{-\alpha^\# n} \le K^\# K_{\mathscr{B}} e^{-\alpha^\#}\left(\|g(\cdot,0)\|_{1,\alpha^\#}+\sum_{s=0}^{n-1} l_g(s)\|\xi(s)\|_{\mathscr{B}}e^{-\alpha^\# s}\right).$$

Then by an application of the discrete Gronwall's inequality [1, Cor. 4.12, p. 183] we get

$$|||\xi|||_{\alpha^\#}K^\# K_{\mathscr{B}} e^{-\alpha^\#}\|g(\cdot,0)\|_{1,\alpha^\#}e^{K^\# K_{\mathscr{B}} e^{-\alpha^\#}\|l_g\|_1}.$$

This ends the proof of the theorem. □

Corollary 4.1. *Assume that the solution operator $T(n)$ of the equation* (1.9) *decays exponentially (see* (3.5)*). In addition, suppose that Condition (D_1) holds with $g(\cdot,0)\in l^1_\alpha$, where α is the constant in* (3.5). *Then there is* *an unique weighted bounded solution y of equation* (1.7) *with $y_0 = 0$ such that $y_\bullet \in l^\infty_\alpha$. Moreover, we have the a priori estimate*

$$\|\| y_\bullet \|\|_{\alpha^\#} K^T K_{\mathscr{B}} e^{-\alpha} \| g(\cdot,0)\|_{1,\alpha^\#} e^{K^T K_{\mathscr{B}} e^{-\alpha} \| l_g \|_1}. \tag{4.4}$$

where K^T and $K_{\mathscr{B}}$ are the constants given by (3.5) *and Axiom (PS$_2$) respectively.*

Next we shall prove that under certain conditions there is a one-to-one correspondence between weighted bounded solutions of (1.2) and its perturbation (1.11).

Theorem 4.2. *Under the conditions of Theorem 1.3, for any solution z of* (1.2) *such that $z_\bullet \in l^\infty_{\alpha^\#}$, there exists an unique solution y of* (1.11) *such that $y_\bullet \in l^\infty_{\alpha^\#}$ and*

$$y_n = z_n + [\Omega y_\bullet](n), \quad n \in \mathbb{Z}^+, \tag{4.5}$$

where Ω is given by (4.1). *Conversely, for any solution y of* (1.11) *there exists an unique solution z of* (1.2) *such that such that $z_\bullet \in l^\infty_{\alpha^\#}$ satisfying* (4.5).

Proof. Let z be any solution of (1.2) such that $z_\bullet \in l^\infty_{\alpha^\#}$. We define the operator Θ on $l^\infty_{\alpha^\#}$ by

$$[\Theta\eta](n) = z_n + \Omega\eta(n), \quad \eta \in l^\infty_{\alpha^\#}, \quad n \geq 0.$$

We claim that $\Theta\eta \in l^\infty_{\alpha^\#}$. Using (4.2), we can assert that

$$\|\|\Theta\eta\|\|_{\alpha^\#} \leq \|\| z_\bullet \|\|_{\alpha^\#} + K^\# K_{\mathscr{B}} e^{-\alpha^\#} \left[\| l_g \|_1 \|\|\eta\|\|_{\alpha^\#} + \| g(\cdot,0)\|_{1,\alpha^\#} \right].$$

We can proceed analogously as in the proof of Theorem 1.3 to obtain

$$\|\| [\Theta^n \xi](n) - [\Theta^n \eta](n) \|\|_{\alpha^\#} \leq \frac{1}{n!} [K^\# K_{\mathscr{B}} e^{-\alpha^\#} \| l_g \|_1]^n \|\| \xi - \eta \|\|_{\alpha^\#}.$$

for any $\xi, \eta \in l^\infty_{\alpha^\#}$. Therefor Θ has a unique fixed point $\xi \in l^\infty_{\alpha^\#}$. Clearly ξ satisfies the relation

$$\xi(n) = z_n + \sum_{s=0}^{n} -1 T(n, s+1) E^0(g(s, \xi(s))).$$

Define

$$y(n) = \begin{cases} [\xi(n)](0) & n \geq 0, \\ [z_0](n), & n < 0. \end{cases}$$

Applying [29, Lemma 2.8] we can see that y is a solution of (1.11) and $y_n = \xi(n)$, $n \in \mathbb{Z}^+$.

Conversely, let y be a solution of (1.11) such that $y_\bullet \in l^\infty_{\alpha^\#}$. From [52, Theorem 2.1] we get

$$y_n = T(n,0) y_0 + [\Omega y_\bullet](n), \quad n \in \mathbb{Z}^+.$$

Define a $\mathscr{B}$-valued function ζ by

$$\zeta(n) = y_n - [\Omega y_\bullet](n), \quad n \in \mathbb{Z}^+.$$

whence $T(n,0)\zeta(0) = \zeta(n)$, $n \in \mathbb{Z}^+$. Set

$$z(n) = \begin{cases} [\zeta(n)](0) & n \geq 0, \\ [\zeta(0)](n), & n < 0. \end{cases}$$

Applying [29, Lemma 2.8] we obtain that z is a solution of (1.2) and $z_n = \zeta(n)$, $n \in \mathbb{Z}^+$. We can verify that $z_\bullet \in l^\infty_{\alpha^\#}$. The proof of the theorem is therefore complete. □

Proof of Theorem 1.4 (page 5). Let z be any solution of (1.2) such that $z_\bullet \in l^\infty_\alpha$. We define the operator $\Theta^\#$ on l^∞_α by

$$\left[\Theta^\# \eta\right](n) = z_n + \sum_{s=0}^{\infty} \Gamma(n,s) E^0(g(s,\eta(s))), \tag{4.6}$$

where $\eta \in l^\infty_{\alpha^\#}$, $n \in \mathbb{Z}^+$ and $\Gamma(n,s)$ denotes the Green function associated with (1.2) (see (2.3).) We have that

$$\begin{aligned} &\left\| \sum_{s=0}^{\infty} \Gamma(n,s) E^0(g(s,\eta(s))) \right\|_{\mathscr{B}} \\ &\quad \leq K_{\mathscr{B}} K_{ed} \sup_{m \in \mathbb{Z}^+} (1 + \|P(m)\|_{\mathscr{B} \to \mathscr{B}}) \\ &\quad\quad \times \left\{ e^\alpha \sum_{s=0}^{n-1} e^{-\alpha n} e^{\alpha s} |g(s,\eta(s))| + e^{-\alpha} \sum_{s=n}^{\infty} e^{\alpha n} e^{-\alpha s} |g(s,\eta(s))| \right\} \\ &\quad \leq e^\alpha e^{\alpha n} K_{\mathscr{B}} K_{ed} \sup_{m \in \mathbb{Z}^+} (1 + \|P(m)\|_{\mathscr{B} \to \mathscr{B}}) \\ &\quad\quad \times \left[\left(\sum_{s=0}^{\infty} l g(s) \right) |||\eta|||_\alpha + \sum_{s=0}^{\infty} |g(s,0)| e^{-\alpha s} \right] \end{aligned}$$

Therefore ??? ANTES O SUP ERA EM $\mathbb{Z}$ E AGORA EH EM $\mathbb{Z}^+$???

$$\begin{aligned} |||\Theta^\# \eta|||_\alpha &\leq |||z_\bullet \\ &|||_\alpha + K_{\mathscr{B}} K_{ed} \sup_{m \geq 0} (1 + \|P(m)\|_{\mathscr{B} \to \mathscr{B}}) \times \left(\|l_g\|_1 \, |||\eta|||_\alpha + \|g(\cdot,0)\|_{1,\alpha} \right). \end{aligned}$$

It proves that the space l^∞_α is invariant under $\Theta^\#$.

A similar argument shows that

$$|||\Theta^\# \eta - \Theta^\# \xi \leq K_{\mathscr{B}} K_{ed} e^\alpha \sup_{m \geq 0} (1 + \|P(m)\|_{\mathscr{B} \to \mathscr{B}}) \|l_g\|_1 \, |||\eta - \Xi|||_\alpha.$$

By (1.13) we have that $\Theta^\#$ is a contraction in l^∞_α and hence $\Theta^\#$ has a unique fixed point $\xi \in l^\infty_\alpha$.

Next we define

$$y(n) = \begin{cases} [\xi(n)](0) & n \geq 0, \\ [\xi(0)](n), & n < 0. \end{cases}$$

Applying [29, Lemma 2.8] we can see that y is a solution of (1.11) and $y_n = \xi(n)$, $n \in \mathbb{Z}^+$. Conversely, let y be a solution of (1.11) such that $y_\bullet \in l_\alpha^\infty$. Define a $\mathscr{B}$-valued function ζ by

$$\zeta(n) = y_n - [\Theta^\# y_\bullet](n), \quad n \in \mathbb{Z}^+.$$

From [52, Theorem 2.1] we can verify that ??? COMEU A EQUACAO ABAIXO, VER P. 98 E 94-95 DO MANUSCRITO ???

$$y_n = T(n,0)y_0 + [\Theta^\# y_\bullet](n), \quad n \in \mathbb{Z}^+.$$

whence $T(n,0)\zeta(0) = \zeta(n)$, $n \in \mathbb{Z}^+$. Set

$$z(n) = \begin{cases} [\zeta(n)](0) & n \geq 0, \\ [\zeta(0)](n), & n < 0. \end{cases}$$

Applying [29, Lemma 2.8] we obtain that z is a solution of (1.2) and $z_n = \zeta(n)$, $n \in \mathbb{Z}^+$.

Next we prove the asymptotic relation (1.14). For $n \geq m$ we have

$$y_n = z_n + \sum_{s=0}^{m-1} T(n,s+1)P(s+1)E^0(g(s,y_s)) + \sum_{s=m}^{\infty} \Gamma(n,s)E^0(g(s,y_s)) \tag{4.7}$$

We can assert that for m sufficiently large

$$\sum_{s=m}^{\infty} \Gamma(n,s)E^0(g(s,y_s)) = o(e^{\alpha n}), \quad n \to \infty. \tag{4.8}$$

In fact, we observe that

$$\begin{aligned} &\left\| \sum_{s=m}^{\infty} \Gamma(n,s)E^0(g(s,y_s)) \right\|_{\mathscr{B}} e^{-\alpha n} \\ &\leq K_{\mathscr{B}} K_{ed} e^{\alpha} \sup_{m \in \mathbb{Z}^+} (1 + \|P(m)\|_{\mathscr{B}\to\mathscr{B}}) \\ &\quad \times \left\{ \left(\sum_{s=m}^{\infty} lg(s) \right) \|\!| y_\bullet |\!\|_\alpha + \sum_{s=m}^{\infty} |g(s,0)| e^{-\alpha s} \right\} \end{aligned} \tag{4.9}$$

On the other hand, we can infer that

$$\left\| \sum_{s=0}^{m-1} T(n,s+1)P(s+1)E^0(g(s,y_s)) \right\|_{\mathscr{B}} e^{-\alpha n} \tag{4.10}$$

$$\leq e^{\alpha(1+2m)}\left\{\left(\sum_{s=0}^{m-1} lg(s)\right)|||y_\bullet|||_\alpha + \sum_{s=0}^{m-1}|g(s,0)|e^{-\alpha s}\right\}.$$

It is clear from expressions (4.7), (4.8) and (4.10) that we get (1.14).

Finally we observe that

$$(1 - K_{\mathscr{B}}K_T e^\alpha \sup_{m\geq 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\|l_g\|_1)|||y_\bullet - \tilde{y}_\bullet|||_\alpha$$
$$\leq |||z_\bullet - \tilde{z}_\bullet|||_\alpha$$
$$\leq (1 + K_{\mathscr{B}}K_T e^\alpha \sup_{m\geq 0}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\|l_g\|_1)|||y_\bullet - \tilde{y}_\bullet|||_\alpha.$$

This establishes the continuity of $y_\bullet \mapsto z_\bullet$ and $z_\bullet \mapsto y_\bullet$. The proof is therefore complete. □

5. LOCAL PERTURBATIONS

Next we will establish a version of Theorem 1.1 which enables us to consider locally Lipschitz perturbations of Equation (1.1).

To state the next result we need to introduce the following assumption.

Condition (F). Suppose that the following conditions hold:

(F$_1$) The function $f\colon \mathbb{Z}^+ \times l^p(\mathbb{Z}^+,\mathscr{B}) \to \mathbb{C}^r$ is locally Lipschitz with respect to the second variable, that is, for each positive number R, for all $n \in \mathbb{Z}^+$ and for all $\xi,\eta \in l^p(\mathbb{Z}^+,\mathscr{B})$ with $\|\xi\| < R$, $\|\eta\| < R$,

$$|f(n,\xi) - f(n,\eta)| \leq l(n,R)\|\xi-\eta\|_p$$

where $l\colon \mathbb{Z}^+ \times [0,\infty)$ is a nondecreasing function with respect to the second variable.

(F$_2$) There is a positive number a such that $\sum_{n=0}^\infty l(n,a) < \infty$.

(F$_3$) $f(\cdot,0) \in l^p(\mathbb{Z}^+,\mathbb{C}^r)$.

We need to introduce some basic notations. We denote by l^p_m the closed subspace of $l^p(\mathbb{Z}^+,\mathscr{B})$ of the sequences $\xi = (\xi(n))$ such that $\xi(n) = 0$ if $0 \leq n \leq m$. For $\lambda > 0$, denote by $l^p_m[\lambda]$ the ball $\|\xi\|_p \leq \lambda$.

Theorem 5.1. *Assume that conditions (A) and (F) are fulfilled and that the functions $N(\cdot)$ and $M(\cdot)$ given by Axiom (PS$_1$) are bounded. In addition suppose that Equation* (1.2) *has an exponential dichotomy on $\mathscr{B}$ with data $(\alpha, K_{ed}, P(\cdot))$. Then there are positive constants $M \in \mathbb{R}$ and $m \in \mathbb{Z}^+$ such that for each $\varphi \in P(m)\mathscr{B}$ with $\|\varphi\|_{\mathscr{B}} \leq M$, there is an unique bounded solution y of Equation* (1.1) *for $n \geq m$ with $P(m)y_m = \varphi$ such that $y_n = 0$ for $0 \leq n \leq m$ and $\|y_\bullet\|_p \leq a$, where a is the constant of Condition (F$_2$). In particular, $y_n = O(1)$??? O ou o??? as $n \to \infty$.*

Proof. Let $\nu \in (0,1)$. Using (F_2) and (F_3) there are n_1 and n_2 in $\mathbb{Z}^+$ such that

$$\frac{2K_{\mathscr{B}}K_{ed}\sup_{n\in\mathbb{Z}^+}(1+\|P(n)\|_{\mathscr{B}\to\mathscr{B}})}{1-e^{-\alpha}}\left[\sum_{j=n_1}^{\infty}|f(j,0)|\right]^{1/p}\le\frac{\nu}{2}a \tag{5.1}$$

and

$$\tau \stackrel{\text{def}}{=} \nu+\frac{2K_{\mathscr{B}}K_{ed}\sup_{n\in\mathbb{Z}^+}(1+\|P(n)\|_{\mathscr{B}\to\mathscr{B}})}{1-e^{-\alpha}}\left[\sum_{j=n_2}^{\infty}|f(j,0)|\right]^{1/p} < 1, \tag{5.2}$$

where $K_{\mathscr{B}}$ is the constant of Axiom (PS_2). Let us denote

$$m \stackrel{\text{def}}{=} \max\{n_1,n_2\},$$
$$M \stackrel{\text{def}}{=} \frac{\nu a(1-e^{-\alpha})}{2K_{\mathscr{B}}K_{ed}\sup_{n\in\mathbb{Z}^+}(1+\|P(n)\|_{\mathscr{B}\to\mathscr{B}})}$$

ans let $\varphi\in P(m)\mathscr{B}$ such that $\|\varphi\|_{\mathscr{B}}\le M$. Let ξ be a sequence in $l^p_{m-1}[a]$. A short argument involving Condition (F) shows that the sequence

$$g_n \stackrel{\text{def}}{=} \begin{cases}0, & 0\le n<m\\ f(n,\xi), & n\ge m,\end{cases} \tag{5.3}$$

belongs to l^p. By discrete maximal regularity (see Theorem 2.4) the Cauchy problem

$$\begin{cases}z(n+1)=L(n,z_n)+g(n), & n\in\mathbb{Z}^+,\\ P(m)z_m=\varphi,\end{cases} \tag{5.4}$$

has an unique solution z such that $\sum_{n=m}^{\infty}\left\|z_n\right\|^p_{\mathscr{B}}<\infty$, which is given by

$$z_n=[\tilde{\mathscr{K}}\xi](n)=T(n,m)P(m)\varphi+\sum_{s=m}^{\infty}\Gamma(n,s)E^0(f(s,\xi)),\quad n\ge m. \tag{5.5}$$

We define $[\tilde{\mathscr{K}}\xi](n)=0$ if $0\le n<m$.

Taking into account Condition (F), we have the following estimates which imply that $\tilde{\mathscr{K}}\xi$ belongs to $l^p_{m-1}[a]$.

$$\begin{aligned}\left[\sum_{n=m}^{\infty}\left\|T(n,m)P(m)\varphi\right\|^p_{\mathscr{B}}\right]^{1/p} \tag{5.6}\\ &\le K_{ed}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\left[\sum_{j=0}^{\infty}e^{-\alpha pj}\right]^{1/p}\|\varphi\|_{\mathscr{B}}\\ &\le K_{ed}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})(1-e^{-\alpha})^{-1}\|\varphi\|_{\mathscr{B}}\end{aligned}$$

$$
\begin{aligned}
(5.7)\quad & \Big[\sum_{n=m}^{\infty}\Big\|\sum_{s=m}^{\infty}\Gamma(n,s)E^0(f(s,0))\Big\|_{\mathscr{B}}^{p}\Big]^{1/p} \\
&\le K_{ed}K_{\mathscr{B}}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}}) \\
&\quad\times\Big[\sum_{n=m}^{\infty}\Big(\sum_{s=m}^{\infty}e^{-\alpha|n-(s+1)|}|f(s,0)|\Big)^{p}\Big]^{1/p} \\
&\le K_{ed}K_{\mathscr{B}}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})\Big(\frac{2}{1-e^{-\alpha}}\Big)^{1/q} \\
&\quad\times\Big[\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}e^{-\alpha|n-(s+1)|}|f(s,0)|^{p}\Big]^{1/p} \\
&\le 2K_{ed}K_{\mathscr{B}}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}}) \\
&\quad\times(1-e^{-\alpha})^{-1}\Big[\sum_{s=m}^{\infty}|f(s,0)|^{p}\Big]^{1/p}.
\end{aligned}
$$

Analogously we have

$$
\begin{aligned}
(5.8)\quad & \Big[\sum_{n=m}^{\infty}\Big\|\sum_{s=m}^{\infty}\Gamma(n,s)E^0\big(f(s,\xi)-f(s,0)\big)\Big\|_{\mathscr{B}}^{p}\Big]^{1/p} \\
&\le 2K_{ed}K_{\mathscr{B}}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}}) \\
&\quad\times(1-e^{-\alpha})^{-1}\Big[\sum_{s=m}^{\infty}l(s,a)^{p}\Big]^{1/p}.
\end{aligned}
$$

Then, inequalities (5.6)–(5.8) together with (5.1) and (5.2) imply

$$
\begin{aligned}
(5.9)\quad \|\tilde{\mathscr{K}}\xi\|_p &\le \frac{K_{ed}}{1-e^{-\alpha}}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}}) \\
&\times\frac{\nu a(1-e^{-\alpha})}{2K_{ed}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})}+\frac{\nu}{2}a+(\tau-\nu)a \\
&=\tau a\le a,
\end{aligned}
$$

proving that $\tilde{\mathscr{K}}\xi$ belongs to $l^p_{m-1}[a]$. On the other hand, for all ξ and η in $l^p_{m-1}[a]$, we obtain that

$$
\begin{aligned}
\|\tilde{\mathscr{K}}\xi-\tilde{\mathscr{K}}\eta\|_p &= \Big[\sum_{n=m}^{\infty}\Big\|\sum_{s=m}^{\infty}\Gamma(n,s)E^0\big(f(s,\xi)-f(s,\eta)\big)\Big\|_{\mathscr{B}}^{p}\Big]^{1/p} \\
&\le 2K_{ed}K_{\mathscr{B}}\sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}}) \\
&\quad\times(1-e^{-\alpha})^{-1}\Big[\sum_{s=m}^{\infty}l(s,a)^{p}\Big]^{1/p}\|\xi-\eta\|_p
\end{aligned}
$$

$$\leq (\tau - \nu)\|\xi - \eta\|_p.$$

Hence $\tilde{\mathscr{K}}$ is a $(\tau - \nu)$-contraction.

Next we will establish the uniqueness of solutions. Let $y = y(n, m, \psi)$ be a solution of (1.1) with the properties stated there. Considering $z(n) = [\tilde{\mathscr{K}} y_\bullet](n)$, it follows from a straight forward computation that

$$z(n) = T(n,m)z(m) + \sum_{s=m}^{n-1} T(n, s+1)E^0(f(s, y_\bullet)), \quad n \geq m.$$

We define a function $a\colon \mathbb{Z} \to \mathbb{C}^r$ by

$$a(n) = \begin{cases} [z(n)](m), & n \geq m, \\ [z(m)](n-m), & n < m. \end{cases}$$

Applying [29, Lemma 2.8], we can infer that $a(n)$ satisfies $a_n = z(n)$, $n \geq m$, and it is a solution of

$$\begin{cases} a(n+1) = L(n, a_n) + f(y, y_\bullet), & n \geq m, \\ P(m)a_m = \varphi. \end{cases}$$

Then the difference $x_n = a_n - y_n$ is a solution of the linear problem (1.2) for $n \geq m$. In this way, we have

$$y_n = [\tilde{\mathscr{K}} y_\bullet](n) + T(n,m)\big([\tilde{\mathscr{K}} y_\bullet](m) - \psi\big).$$

Now, putting $\Phi(n) \overset{\text{def}}{=} \big\|T(n,m)Q(m)([\tilde{\mathscr{K}} y_\bullet](m) - \psi)\big\|_{\mathscr{B}}^{-1}$ and $\Psi(n) \overset{\text{def}}{=} \sum_{s=n}^{\infty} \Phi(s+1)$, we have

$$\begin{aligned} \frac{\Psi(n)}{\Phi(n)} &= \sum_{s=n}^{\infty} \big\|T(n,m)Q(m)([\tilde{\mathscr{K}} y_\bullet](m) - \psi)\big\|\Phi(s+1) \\ &\leq K_{ed} \sum_{s=n}^{\infty} e^{\alpha(n-(s+1))}\|Q(s+1)\|_{\mathscr{B}\to\mathscr{B}}\Phi(s+1)^{-1}\Phi(s+1)??? \\ &\leq K_{ed} \sup_{m\in\mathbb{Z}^+} (1 + \|P(m)\|_{\mathscr{B}\to\mathscr{B}})(1 - e^{-\alpha})^{-1}, \end{aligned}$$

hence $\Psi(n) \leq K_{ed} \sup_{m\in\mathbb{Z}^+}(1+\|P(m)\|_{\mathscr{B}\to\mathscr{B}})(1-e^{-\alpha})^{-1}\Phi(n)$, for all $n \geq m$, which means that $\big\{\|T(n,m)Q(m)([\tilde{\mathscr{K}} y_\bullet](m) - \psi)\|_{\mathscr{B}}\big\}_{n\geq m}$ is unbounded. Since

$$T(n,m)([\tilde{\mathscr{K}} y_\bullet](m) - \psi) = T(n,m)Q(m)([\tilde{\mathscr{K}} y_\bullet](m) - \psi)$$

and $y_\bullet$, $\tilde{\mathscr{K}} y_\bullet$ are bounded, we conclude that $y_n = [\tilde{\mathscr{K}} y_\bullet](n)$. Hence the uniqueness of y follows from the uniqueness of the fixed point of the map $\tilde{\mathscr{K}}$. This completes the proof of the theorem. □

Now we establish a local version of the Theorem 1.3. To state such version, we will require the following assumption.

Condition (G). Suppose that the following conditions hold:

(G_1) The function $g\colon \mathbb{Z}^+ \times \mathscr{B} \to \mathbb{C}^r$ is locally Lipschitz with respect to the second variable, that is, for each positive number R, for all $n \in \mathbb{Z}^+$ and for all $\varphi, \psi \in \mathscr{B}$ with $\|\varphi\| \le R$, $\|\psi\| \le R$, we get

$$|g(n,\varphi) - g(n,\psi)| \le \rho(n,R)\|\varphi - \psi\|_{\mathscr{B}},$$

where $\rho\colon \mathbb{Z}^+ \times [0,\infty)$ is a nondecreasing function with respect to the second variable.

(G_2) There is a positive number a such that

$$\sum_{n=0}^{\infty} \rho(n, ae^{\alpha^\# n}) < \infty.$$

(G_3) $g(\cdot, 0) \in l^1_{\alpha^\#}(\mathbb{Z}^+, \mathbb{C}^r)$.

We have the following result.

Theorem 5.2. *Assume that conditions (A) and (G) are fulfilled and that the functions $N(\cdot)$ and $M(\cdot)$ given by Axiom (PS$_1$) are bounded. Then there is an unique weighted bounded solution y of Equation* (1.11) *for $n > m$ with $y_m = 0$ such that $\|y_m\|_{\mathscr{B}}\, e^{-\alpha^\# n} \le a$, where a is the constant of Condition (G$_2$).*

Proof. Let $\nu \in (0,1)$. Using (G$_2$) and (G$_3$) there are n_1 and n_2 in $\mathbb{Z}^+$ such that

$$K_{\mathscr{B}} K^\# e^{-\alpha^\#} \sum_{s=n_1}^{\infty} |g(s,0)| e^{-\alpha^\# s} \le \nu a, \tag{5.10}$$

and

$$\tau \stackrel{\text{def}}{=} \nu + K_{\mathscr{B}} K^\# e^{-\alpha^\#} \sum_{s=n_2}^{\infty} l(s, ae^{-\alpha^\# s}) < 1, \tag{5.11}$$

where $K^\#$ and $\alpha^\#$ are the constants of Theorem 2.5. Set $m = \max\{n_1, n_2\}$.

Denote by $\mathscr{L}^\infty_\#$ the Banach space of all weighted bounded functions $\xi\colon \mathbb{N}(m+1) \to \mathscr{B}$??? QUE NOTACAO EH ESSA ??? equipped with the norm $|||\xi|||_{\alpha^\#} = \sup_{n \ge m+1} \|\xi(n)\|_{\mathscr{B}} e^{-\alpha^\# n}$. Denote by $\mathscr{L}^\infty_\#[a]$ the ball $|||\xi|||_{\alpha^\#} \le a$ in $\mathscr{L}^\infty_\#$.

We define the operator Ω on $\mathscr{L}^\infty_\#[a]$ by

$$[\Omega\xi](n) = \sum_{s=m}^{n-1} T(s, s+1) E^0(g(s, \xi(s))). \tag{5.12}$$

We observe that Ω is well defined. In fact,

$$\begin{aligned} &\|[\Omega\xi](n)\|_{\mathscr{B}} e^{-\alpha^\# n} \\ &\qquad \le K_{\mathscr{B}} K^\# e^{-\alpha^\#} \sum_{s=m}^{\infty} e^{-\alpha^\# s} |g(s, \xi(s))| \end{aligned}$$

$$\leq K_{\mathscr{B}} K^{\#} e^{-\alpha^{\#}} \sum_{s=m}^{n-1} e^{-\alpha^{\#} s}\big(l(s, ae^{-\alpha^{\#} s}) \|\xi(s)\|_{\mathscr{B}} + |g(s,0)|\big)$$
$$\leq (\tau - \nu)a + \nu a \leq \tau a < a,$$

whence $|||\omega\xi|||_{\alpha^{\#}} \leq a$.

Let ξ and η in $\mathscr{L}^{\infty}_{\#}[a]$. We have

$$\begin{aligned} &\|[\omega\xi](n) - [\omega\eta](n)\|_{\mathscr{B}} e^{-\alpha^{\#} n} \\ &\qquad \leq K_{\mathscr{B}} K^{\#} e^{-\alpha^{\#}} \sum_{s=m}^{n-1} l(s, ae^{-\alpha^{\#} s}) \|\xi(s) - \eta(s)\|_{\mathscr{B}} e^{-\alpha^{\#} s} \\ &\qquad \leq K_{\mathscr{B}} K^{\#} e^{-\alpha^{\#}} \sum_{s=m}^{\infty} l(s, ae^{-\alpha^{\#} s}) |||\xi(s) - \eta(s)|||_{\alpha^{\#}} \\ &\qquad \leq (\tau - \nu) |||\xi(s) - \eta(s)|||_{\alpha^{\#}}. \end{aligned}$$

Hence Ω is a $(\tau - \nu)$-contraction. This completes the proof of the theorem. □

Theorem 5.3. *Under the conditions of Theorem 5.2, there are* $\epsilon > 0$ *and* $m \in \mathbb{Z}^{+}$ *so that for any solution* $z(n)$ *of* (1.2) *for* $n \geq m$ *such that* $z_{\bullet} \in \mathscr{L}^{\infty}_{\#}[\epsilon a]$ *(see proof of Theorem 5.2 for the definition of* $\mathscr{L}^{\infty}_{\#}[\epsilon a]$*,) there exists an unique solution* $y(n)$ *of* (1.11) *for* $n \geq m$ *such that* $y_{\bullet} \in \mathscr{L}^{\infty}_{\#}[a]$ *and* (4.5)*,* $n \geq m$*, holds with* Ω *defined by* (4.1)*. Conversely, for any solution* $y(n)$ *of* (1.11) *for* $n \geq m$ *such that* $y_{\bullet} \in \mathscr{L}^{\infty}_{\#}[\epsilon a]$*, there exists an unique solution* $z(n)$ *of* (1.2) *for* $n \geq m$ *such that* $z_{\bullet} \in \mathscr{L}^{\infty}_{\#}[a]$ *satisfying* (4.5) *for* $n \geq m$*.*

Theorem 5.4. *Assume that Condition (A) is fulfilled and that the functions* $N(\cdot)$ *and* $M(\cdot)$ *given by Axiom (PS$_1$) are bounded. Furthermore, suppose that Equation* (1.2) *has an exponential dichotomy with data* $(\alpha, K_{ed}, P(\cdot))$ *and assume that Condition (F) holds with* α *instead of* $\alpha^{\#}$*. Then there are positive constants* $\epsilon \in \mathbb{R}$ *and* $m \in \mathbb{Z}^{+}$ *so that for any solution* $z(n)$ *of* (1.2) *for* $n \geq m$ *such that* $z_{\bullet} \in \mathscr{L}^{\infty}_{\alpha}[\epsilon a]$*, there exists an unique solution* $y(n)$ *of* (1.11) *for* $n \geq m$ *such that* $y_{\bullet} \in \mathscr{L}^{\infty}_{\alpha}[a]$ *and the asymptotic relation* (1.14) *holds. The conversely is also true. Furthermore, the one-to-one correspondences* $y_{\bullet} \mapsto z_{\bullet}$ *and* $z_{\bullet} \mapsto y_{\bullet}$ *are continuous. ??? AQUI COLOQUEI CORRESPONDENCIAS ENTRE Y E Z, E NAO Y E X, COMO NO MANUSCRITO ???*

6. Asymptotic periodicity

Proof of Theorem 1.5. We define the operator Γ on the space $\mathrm{SAP}_\omega(\mathscr{B})$ (see Definition 2.7) by (3.3); we write $[\Gamma\xi](n) = T(n)\varphi + v_\xi(n)$, where

$$v_\xi(n) \overset{\text{def}}{=} \sum_{s=0}^{n-1} T(n-1-s)E^0(g(s,\xi(s))), \quad \xi \in \mathrm{SAP}_\omega(\mathscr{B}). \tag{6.1}$$

We shall prove that Γ is well defined. It is easy to see that the remains of the proof is the same as in Theorem 1.2. We note that the function $T(\cdot)\varphi \in \mathrm{SAP}_\omega(\mathscr{B})$.

Moreover, in view of that the semigroup $T(n)$ is uniformly bounded in $\mathbb{Z}^+$, there is a constant $K_T \geq 1$ such that $\|T(n)\|_{\mathscr{B}\to\mathscr{B}} \leq K_T$ for all $n \in \mathbb{Z}^+$. We get the following estimate

$$\|v_\xi\|_\infty \leq K_T K_{\mathscr{B}}\big[\|l_g\|_1\,\|\xi\|_\infty + \|g(\cdot,0)\|_1\big].$$

On the other hand, we have the following estimates

$$\begin{aligned}\left\|\sum_{s=n_1}^{m} T(m-s)E^0(g(s,\xi(s)))\right\| &\leq K_T \sum_{s=n_1}^{m} |g(s,\xi(s))| \\ &\leq K_T \sum_{s=n_1}^{m} \big(l_g(s)|\xi(s)| + |g(s,0)|\big) \\ &\leq K_T\left[\left(\sum_{s=n_1}^{\infty} l_g(s)\right)\|\xi\|_\infty + \left(\sum_{s=n_1}^{\infty} |g(s,0)|\right)\right].\end{aligned}$$

Hence we obtain that

$$\lim_{???} \sum_{???}^{m} T(m-s)E^0(g(s,\xi(s))) = 0 \quad \text{uniformly in } m.$$

Taking into account that $T(\cdot)$ is S-asymptotically ω-periodic and the above property, we infer from the decomposition

$$\begin{aligned}v_\xi(n+\omega) - v_\xi(n) = &\sum_{s=0}^{n_1-1}\big[T(n-1-s+\omega) - T(n-1-s)\big]E^0(g(s,\xi(s))) \\ &+ \sum_{s=n_1}^{n_1-1+\omega} T(n-1-s+\omega)E^0(g(s,\xi(s))) \\ &- \sum_{s=n_1}^{n_1-1} T(n-1-s)E^0(g(s,\xi(s))),\end{aligned}$$

that

$$v_\xi(n+\omega) - v_\xi(n) \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Remark 6.1. A result similar to Theorem 1.5 was obtained by Henríquez *et al.* [45] for first order abstract Cauchy problem in Banach spaces.

Notice that Theorem 1.5 does not cover the cases when the perturbation g of (1.7) has a Lipschitz factor l_g being a constant. Next we establish a new result on the existence of discrete S-asymptotically ω-periodic solutions of problem (1.7)–(1.8).

Theorem 6.2. *Assume that the solution operator $T(n)$ of Equation (1.9) decays exponentially (see (1.15), page 7.) Let Condition (E)*[3] *be satisfied and assume that g is uniformly S-asymptotically ω-periodic on bounded sets. If*

$$K_{\mathscr{B}}K^T K_g + e^{-\alpha} < 1$$

??? NO FINAL DA PROVA DIZ QUE Γ EH UMA $[K_{\mathscr{B}}K^T K_g(1-e^{-\alpha})^{-1}]$-CONTRACTION. NAO SERIA ESSE NUMERO NA DESIGUALDADE ACIMA ???
then there is a unique discrete S-asymptotically ω-periodic solution of problem (1.7)–(1.8).

Proof. We define the map Γ on the space $\mathrm{SAP}_\omega(\mathscr{B})$ by the expression $[\Gamma\xi](n) = T(n)\varphi + v_\xi(n)$, where $\xi \in \mathrm{SAP}_\omega(\mathscr{B})$ and v_ξ is given by (6.1).

We will show initially that Γ is $\mathrm{SAP}_\omega(\mathscr{B})$-valued. Since $T(n)$ decays exponentially, the problem is reduced to show that v_ξ belongs to $\mathrm{SAP}_\omega(\mathscr{B})$. Using the fact that $g(\cdot,\xi(\cdot))$ is a bounded function, it follows that $v_\xi \in l^\infty(\mathbb{Z}^+,\mathscr{B})$. On the other hand, in view that g is asymptotically uniformly continuous on bounded sets and applying Lemma 2.11, we have that for each $\epsilon > 0$ there is a constant $N_\epsilon \in \mathbb{Z}^+$ so that

$$|g(n+\omega,\xi(n+\omega)) - g(n,\xi(n))| < \epsilon, \quad n > N_\epsilon.$$

For $n > N_\epsilon$ we estimate

$$\begin{aligned}\| v_\xi(n+\Omega) - v_\xi(n)\|_{\mathscr{B}} & \\ &\le K_{\mathscr{B}}K^T(1-e^{-\alpha})^{-1}\|g(\cdot,\xi(\cdot))\|_\infty e^{-\alpha n} \\ &\quad + 2K_{\mathscr{B}}K^T\|g(\cdot,\xi(\cdot))\|_\infty \sum_{s=n-N_\epsilon}^{\infty} e^{-\alpha s} \\ &\quad + K_{\mathscr{B}}K^T(1-e^{-\alpha})^{-1}\epsilon,\end{aligned}$$

which permit us to infer that $v_\xi(n+\Omega) - v_\xi(n) \to 0$ as $n \to \infty$. This shows that $\Gamma\xi \in \mathrm{SAP}_\omega(\mathscr{B})$. Pretty easy calculations show that the operator Γ is a $[K_{\mathscr{B}}K^T K_g(1-e^{-\alpha})^{-1}]$-contraction, which finishes the proof of Theorem 6.2. □

[3] A CONDICAO (E) EH A MESMA QUE A SEGUNDA CONDICAO (G), NA P 123 DO MANUSCRITO.

Proof of Theorem 1.6. Let $\mathscr{S}(\mathscr{B})$ the the space consisting of all sequences $\xi \in l^\infty(\mathbb{Z}^+,\mathscr{B})$ such that $\lim_{n\to\infty}\xi(n+m\omega)-\xi(n)=0$ uniformly for $m\in\mathbb{Z}^+$. It is easy to see that $\mathscr{S}(\mathscr{B})$ is a closed subspace of $l^\infty(\mathbb{Z}^+,\mathscr{B})$. Let $\xi\in\mathscr{S}(\mathscr{B})$. Since Range($\xi$) is a bounded set and taking into account that $\lim_{n\to\infty} g(n+m\omega,\varphi)-g(n,\varphi)=0$ uniformly for $\varphi\in$ Range(ξ), it follows that

$$\lim_{n\to\infty} g(n+m\omega,\xi(n+m\omega))-g(n,\xi(n))=0$$

uniformly for $m\in\mathbb{Z}^+$. We keep notations introduced in the proof of Theorem 1.5. We consider the map Γ defined in $\mathscr{S}(\mathscr{B})$. From

$$\begin{aligned}\|v_\xi\|_\infty &\le K_{\mathscr{B}}K^T(1-e^{-\alpha})^{-1}\big[K_g\|\xi\|_\infty+\|g(\cdot,0)\|_\infty\big],\\ \|v_\xi(n+m\omega)-v_\xi(n)\|_{\mathscr{B}} &\le K_{\mathscr{B}}K^T(1-e^{-\alpha})^{-1}\epsilon\\ &\quad+K_{\mathscr{B}}K^T\big[K_g\|\xi\|_\infty+\|g(\cdot,0)\|_\infty\big]\\ &\quad\times\left(\sum_{j=n}^{\infty}e^{-\alpha j}+2\sum_{j=n-N_\epsilon}^{\infty}e^{-\alpha j}\right),\quad n>N_\epsilon,\end{aligned}$$

we get that Γ is $\mathscr{S}(\mathscr{B})$-valued. Therefore the fixed point of Γ belongs to $\mathscr{S}(\mathscr{B})$ and the assertion is consequence of [45, Cor. 3.1]. The proof is complete. □

7. Applications to Volterra difference systems

We complete this work by applying our previous results to the Volterra difference systems with infinite delay. Let γ be a positive real number and let $A(n)$, $K(n)$ and $D(n,s)$ be three $r\times r$ matrices defined for $n\in\mathbb{Z}^+$, $s\in\mathbb{Z}$ such that

$$\|A\|_\infty=\sup n\ge 0|A(n)|<\infty \tag{7.1}$$

and

$$\sum_{n=0}^{\infty}|K(n)|e^{\gamma n}<\infty. \tag{7.2}$$

We consider the following Volterra difference systems with infinite delay.

$$x(n+1)=\sum_{s=-\infty}^{n}A(n)K(n-s)x(s),\quad n\ge 0, \tag{7.3}$$

$$y(n+1)=\sum_{s=-\infty}^{n}\big[A(n)K(n-s)+\nu D(n,s)\big]y(s),\quad n\ge 0, \tag{7.4}$$

where ν is a real number.

We recall that Volterra system (7.3) and (7.4) are viewed as a retarded functional difference equations on the phase space $\mathscr{B}_\gamma$, where $\mathscr{B}_\gamma$ is defined as in (1.3). We have, as consequences of Theorem 1.1, the following result.

Theorem 7.1. *Suppose that the following hypothesis hold:*

(i) *System* (7.3) *possesses an exponential dichotomy.*

(ii) *There is a sequence $\beta \in l^p(\mathbb{Z}^+)$ such that*

$$\sum_{\tau=0}^{n} |D(n,\tau)| + \sum_{\tau=-\infty}^{-1} |D(n,\tau)| e^{-\gamma\tau} \le \beta(n), \quad n \ge 0.$$

If $|\nu|$ is small enough, then for each $\varphi \in P(0)\mathscr{B}$ there is a unique bounded solution y of the System (7.4) *with $P(0)y_0 = \varphi$ such that $y_\bullet \in l^p(\mathbb{Z}^+, \mathscr{B}_\gamma)$, in particular $y_\bullet \in l^p(\mathbb{Z}^+, \mathbb{C}^r)$. Moreover, we have the following a priori estimate for the solution:*

$$\| y_\bullet \|_p \le C \|\varphi\|_{\mathscr{B}_\gamma},$$

where $C > 0$ is a suitable constant. Furthermore, the application $\varphi \in P(n_0)\mathscr{B}_\gamma \mapsto y_\bullet(\varphi) \in l^p(\mathbb{Z}^+, \mathscr{B}_\gamma)$ is continuous.

Let $B(n)$ and $G(s)$ be two $r \times r$ matrices defined for $n \in \mathbb{Z}^+$ and $s \in \mathbb{Z}^-$ such that $|B(\cdot)| \in l^p$ and

$$\sum_{n=0}^{\infty} |G(-n)| e^{\gamma n} < \infty. \tag{7.5}$$

Next we consider the following Volterra difference system with infinite delay:

$$y(n+1) = \sum_{s=-\infty}^{n} \left[A(n)K(n-s) + B(n)G(s-n)|y(0)| \right] y(s), \quad n \ge 0. \tag{7.6}$$

As a consequence of Theorem 5.1 we have the following result.

Theorem 7.2. *Suppose that System* (7.3) *has an exponential dichotomy. Then there are positive constants $M \in \mathbb{R}$ and $m \in \mathbb{Z}^+$ such that for each $\varphi \in P(m)\mathscr{B}_\gamma$ with $\|\varphi\|_{\mathscr{B}_\gamma} \le M$, there is a unique bounded solution y of the Volterra system* (7.6) *for $n \ge m$ with $P(m)y_m = \varphi$ such that $y_n = o(1)$ as $n \to \infty$.*

Example 7.3. Let $a_i(n)$, $i = 1,2$, be two bounded sequences in $\mathbb{Z}^+$ and σ, α, γ be three positive constants such that

(i) $\rho_1^* = \sup_{n \ge 0} \max_{-n \le \theta \le 0} \left[\prod_{s=n+\theta}^{n-1} |a_1(s)|^{-1} e^{\gamma\theta} \right] < \infty,$

(ii) $\prod_{s=\tau}^{n-1} |a_1(s)| \le \sigma e^{-\alpha(n-\tau)}, \quad n \ge \tau \ge 0,$

(iii) $\prod_{s=n}^{\tau-1} |a_2(s)|^{-1} \le \sigma e^{-\alpha(\tau-n)}, \quad \tau \ge n \ge 0.$

Set

$$\rho_2^* = \sup_{n\ge 0} \max_{-n\le\theta\le 0} \left[\prod_{s=n+\theta}^{n-1} |a_2(s)|^{-1} e^{\gamma\theta} \right].$$

We consider the following nonautonomous difference equation

$$x(n+1) = L(n, x_n), \quad n \ge 0, \tag{7.7}$$

with $L(n,\varphi) = A(n)\varphi(0)$, $\varphi \in \mathscr{B}_\gamma$, where $A(n) = \mathrm{diag}\big(a_1(n), a_2(n)\big)$. The solution operator $T(n\tau)$, $n \ge \tau$, of (7.7) is a bounded linear operator on the phase space $\mathscr{B}_\gamma$ given by

$$[T(n,\tau)\varphi](\theta) = \begin{cases} \left(\varphi^1(0) \prod_{s=\tau}^{n+\theta-1} a_1(s), \varphi^2(0) \prod_{s=\tau}^{n+\theta-1} a_2(s)\right), & -(n-\tau) \le \theta \le 0, \\ \big(\varphi^1(n-\tau+\theta), \varphi^2(n-\tau+\theta)\big), & \theta < -(n-\tau). \end{cases}$$

We define the projections $P(n), Q(n) \colon \mathscr{B}_\gamma \to \mathscr{B}_\gamma$ by

$$[P(n)\varphi](\theta) \quad \begin{cases} \left(\varphi^1(\theta), \varphi^2(\theta) - \varphi^2(0) \prod_{s=n+\theta}^{n-1} \frac{1}{a_2(s)},\right) & -n \le \theta \le 0, \\ \big(\varphi^1(\theta), \varphi^2(\theta)\big), & \theta < -n, \end{cases}$$

and

$$[Q(n)\varphi](\theta) \quad \begin{cases} \left(0, \varphi^2(0) \prod_{s=n+\theta}^{n-1} \frac{1}{a_2(s)},\right) & -n \le \theta \le 0, \\ (0,0), & \theta < -n. \end{cases}$$

We can prove that $T(n,\tau)$,, $n \ge \tau$, is an isomorphism from $Q(\tau)\mathscr{B}_\gamma$ onto $Q(n)\mathscr{B}_\gamma$. The inverse mapping is given by

$$[T(\tau,n)Q(n)\varphi](\theta) \quad \begin{cases} \left(0, \varphi^2(0) \prod_{s=\tau+\theta}^{n-1} \frac{1}{a_2(s)},\right) & -\tau \le \theta \le 0, \\ (0,0), & \theta < -\tau. \end{cases}$$

We have the following estimates:

$$\| T(n,\tau)P(\tau) \|_{\mathscr{B}_\gamma \to \mathscr{B}_\gamma} \le 4\sigma^2 \rho_1^* e^{-\alpha(n-\tau)}, \quad n \ge \tau, \tag{7.8}$$

$$\| T(n,\tau)Q(\tau) \|_{\mathscr{B}_\gamma \to \mathscr{B}_\gamma} \le \sigma \rho_2^* e^{-\alpha(\tau-n)}, \quad \tau \ge n. \tag{7.9}$$

From estimates (7.8) and (7.9) we get that Equation (7.7) has an exponential dichotomy with data $(\alpha, K, P(\cdot))$ where $K = 4\sigma^2\rho_1^* + \sigma\rho_2^*$.

Next we consider the following perturbation of Equation (7.7):

$$x(n+1) = L_1(n, x_n), \quad n \geq 0, \tag{7.10}$$

where $L_1(n\varphi) = L(n,\varphi)+B(n)\varphi(-1)$ and $B(n)$ is a 2×2 matrix with $\|B(\cdot)\|_\infty$ sufficiently small. Then by the Cardoso-Cuevas' perturbation theorem [13, Theorem 1.3] Equation (7.10) has an exponential dichotomy for suitable data $(\tilde{\alpha}, \tilde{K}, \tilde{P}(\cdot))$.

Let $D(n)$ be a 2×2 matrix defined for $n \in \mathbb{Z}^+$ such that $\sum_{n=0}^{\infty} |D(n)|^p < \infty$. We have, as consequence of Theorem (7.1), the following result.

Proposition 7.4. *Let ν be a real number such that $|\nu|$ is small enough, and $\varphi \in \tilde{P}(0)\mathscr{B}_\gamma$. Then*

$$x(n+1) = L_1(n, x_n) + \nu D(n)x(n), \quad n \geq 0, \tag{7.11}$$

has a unique bounded solution $y(n)$ with $\tilde{P}(0)y_0 = \varphi$ such that $y \in l^p(\mathbb{Z}^+, \mathbb{C}^2)$.

Next we consider the equation

$$x(n+1) = L_1(n, x_n) + D(n)|x(0)|x(n), \quad n \geq 0. \tag{7.12}$$

By Theorem (5.1), there are positive constants $M \in \mathbb{R}$ and $m \in \mathbb{Z}^+$ such that for each $\varphi \in \tilde{P}(m)\mathscr{B}_\gamma$ with $\|\varphi\|_{\mathscr{B}_\gamma} \leq M$, there is a unique bounded solution y of (7.12) for $n \geq m$ with $\tilde{P}(m)y_m = \varphi$ such that the map $n \mapsto y_n$ belongs to $l^p(\mathbb{Z}^+, \mathscr{B}_\gamma)$. This finishes the discussion of Example 7.3.

Example 7.5. Let $K(n)$ be an $r \times r$ matrix defined for $n \in \mathbb{Z}^+$ such that $K \in l^1_{-\gamma}$ (see Section 2 for the definition of $l^1_{-\gamma}$.) We consider the linear Volterra difference equation on $\mathbb{C}^r$:

$$x(n+1) = \sum_{s=-\infty}^{n} K(n-s)x(s), \quad n \geq 0. \tag{7.13}$$

Let $\tilde{K}(z)$ be a z-transform of $K(n)$ (*e. g.* see [34].) Notice that $\tilde{K}(z)$ exists and is analytic in the domain $|z| > e^{-\gamma}$ on the complex plane. Suppose that the characteristic operator of (7.13), $zI - \tilde{K}(z)$, is invertible for $|z| \geq 1$, that is, $\left(zI - \tilde{K}(z)\right)^{-1} \in \mathscr{L}(\mathbb{C}^r)$ for all $|z| \geq 1$, where I is the identity operator on $\mathbb{C}^r$. Let $T(n)$ be the solution operator of (7.13) in $\mathscr{B}_\gamma$. Then by Furumochi *et al.* [36, Corollary 2.3] $T(n)$ decays exponentially (see (1.15).) From Theorem 6.2 we obtain the following result.

Proposition 7.6. *Let $a\colon \mathbb{Z}^+ \to \mathbb{R}$ be a S-asymptotically ω-periodic sequence. Let ν be a real number such that $|\nu|$ is sufficiently small and $\varphi \in \mathscr{B}_\gamma$. Then*

$$x(n+1) = \sum_{s=\infty}^{n} K(n-s)x(s) + \nu a(n)x(n), \quad n \geq 0,$$

has a unique solution $x(n)$ which is discrete S-asymptotically periodic such that $x_0 = \varphi$.

This finishes the discussion of Example 7.5.

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