Acta Mathematicae Applicatae Sinica, English Series
Vol. 18, No. 4 (2002) 553–560

Acta Mathematicae Applicatae Sinica, English Series

Linear Volterra Integral Equations

M. Federson[1], **R. Bianconi**[2], **L. Barbanti**[3]

[1]Department of Mathematics, University of São Paulo, CP 668, 13560-970 SP, Brazil
(E-mail: federson@icmc.sc.usp.br)

[2,3]Department of Mathematics, University of São Paulo, CP 66281, 05315-970 SP, Brazil
(E-mail: [2]bianconi@ime.usp.br; [3]barbanti@ime.usp.br)

Abstract The Kurzweil-Henstock integral formalism is applied to establish the existence of solutions to the linear integral equations of Volterra-type

$$x(t) + {}^{*}\!\!\int_{[a,t]} \alpha(s)x(s)\,ds = f(t), \qquad t \in [a,b], \tag{1}$$

where the functions are Banach-space valued. Special theorems on existence of solutions concerning the Lebesgue integral setting are obtained. These sharpen earlier results.

Keywords Linear Volterra integral equations, Kurzweil-Henstock integrals
2000 MR Subject Classification 45A05, 26A39, 28B05

1 Introduction

In [2], we considered the abstract linear integral equation of Volterra-Henstock

$$x(t) + {}^{K}\!\!\int_{[a,t]} \alpha(s)x(s)\,ds = f(t), \qquad t \in [a,b], \tag{2}$$

as the limit of the following linear integral equations of Volterra-Bochner-Lebesgue

$$x(t) + {}^{L}\!\!\int \_X_n \cap [a,t]\alpha(s)x(s)\,ds = f(t), \qquad t \in [a,b], \quad n \in \mathbb{N}, \tag{3}$$

where $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of closed sets such that $X_n \uparrow [a,b]$ (i.e., $X_n \subset X_{n+1} \subset [a,b]$ for every $n \in \mathbb{N}$, and $\cup X_n = [a,b]$) and ${}^{K}\!\!\int$ and ${}^{L}\!\!\int$ denote respectively the Henstock and the Bochner-Lebesgue integrals. On that occasion, we supposed that either α was a bounded Henstock integrable function (possibly highly oscillating) and x, f were functions of bounded variation (with discontinuities of the first kind), or α was a Henstock absolutely integrable function (Lebesgue integrable in the real case) and x, f were continuous functions. Then the limit of solutions of (3) was a solution of (2), provided the limit existed and either α was smaller than 1 in absolute value or the integral of $\|\alpha(\cdot)\|$ was smaller than 1.

In the present paper, we improve the result above by lifting this last hypothesis on α. We transform (3) into the linear Stieltjes integral equation

$$x(t) + {}^{L}\!\!\int_{X_n\cap[a,t]} d\tilde{\alpha}(s)x(s)\,ds = f(t), \qquad t \in [a,b], \quad n \in \mathbb{N} \tag{4}$$

Manuscript received July 24, 2000.

where $\tilde{\alpha}$ denotes the indefinite integral of α, by integration by parts and using a result on the existence of solutions of each (4) due to [1]. Then the procedure of [2] is applied and the limit of solutions of (3) satisfies equation (2). Special results are given when we consider the Bochner-Lebesgue integral in (2).

Let $[a,b]$ be a compact interval of the real line $\mathbb{R}$.

Let X and Y be Banach spaces, $L(X,Y)$ be the Banach space of all linear continuous functions from X to Y, $L(X)=L(X,X)$ and $X'=L(X,\mathbb{R})$. Let $C\big([a,b],X\big)$ and $G\big([a,b],X\big)$ be respectively the Banach spaces of continuous and of regulated functions from $[a,b]$ to X endowed with the supremum norm, $\|\cdot\|_\infty$.

Given a function $\alpha:[a,b]\to L(X,Y)$ and $x\in X$, consider the function $\alpha x: t\in[a,b]\to \alpha(t)x\in Y$. We say that α is weakly continuous (respectively weakly regulated) and we write $\alpha\in C^\sigma\big([a,b],L(X,Y)\big)$ $\big($resp. $\alpha\in G^\sigma\big([a,b],L(X,Y)\big)\big)$ if for every $x\in X$, the function αx is continuous (resp. αx is regulated). Let $G^-\big([a,b],X\big)$ be the set of all $f\in G\big([a,b],X\big)$ such that f is left continuous and let $G^{\sigma-}\big([a,b],L(X,Y)\big)$ be the set of all $\alpha\in G^\sigma\big([a,b],L(X,Y)\big)$ such that for every $x\in X$, the function αx is left continuous. In an analogous way, we define $G^+([a,b],X)$ and $G^{\sigma+}\big([a,b],L(X,Y)\big)$ for the right continuity.

Let $d=(t_i)$ be a division of $[a,b]$ (i.e., $a=t_0<t_1<\cdots<t_n=b$). We write $d=(t_i)\in D_{[a,b]}$ and $|d|=n$. Given $d=(t_i)\in D_{[a,b]}$ and functions $\alpha:[a,b]\to L(X,Y)$ and $f:[a,b]\to X$, we define

$$V_d(f)=\sum_i \big\|f(t_i)-f(t_{i-1})\big\|,$$

$$SV_d(\alpha)=\sup\left\{\Big\|\sum_i \big[\alpha(t_i)-\alpha(t_{i-1})\big]y_i\Big\|;\ y_i\in Y,\ \|y_i\|\le 1\right\}.$$

Then $V(f)=\sup\big\{V_d(f);\ d\in D_{[a,b]}\big\}$ is the variation of f and $SV(\alpha)=\sup\big\{SV_d(\alpha);\ d\in D_{[a,b]}\big\}$ is the semivariation of α. If $V(f)<\infty$, then f is of bounded variation and we write $f\in BV([a,b],X)$. If $SV(\alpha)<\infty$, then α is of bounded semivariation and we write $\alpha\in SV\big([a,b],L(X,Y)\big)$. Clearly $BV\big([a,b],L(X,Y)\big)\subset SV\big([a,b],L(X,Y)\big)$. Besides, $SV\big([a,b],L(X,\mathbb{R})\big)=BV\big([a,b],X'\big)$ and, if X is of finite dimension, then $SV\big([a,b],L(X)\big)=BV\big([a,b],L(X)\big)$. Under the norm given by the variation, the following spaces are complete:

$$BV_a\big([a,b],X\big)=\big\{f\in BV\big([a,b],X\big);\ f(a)=0\big\},$$
$$BV_a^-\big([a,b],X\big)=\big\{f\in BV_a\big([a,b],X\big);\ f \text{ is left continuous}\big\}.$$

For more information on the spaces above, see [5].

2 Kurzweil and Henstock Vector Integrals

2.1 Definitions

We say that $d=(\xi_i,t_i)$ is a **tagged division** of $[a,b]$, if $d=(t_i)\in D_{[a,b]}$ and $\xi_i\in[t_{i-1},t_i]$, for $i=1,2,\cdots,|d|$. Then $TD_{[a,b]}$ is the set of all tagged divisions of $[a,b]$. Given a function $\delta:[a,b]\to]0,\infty[$ (called a **gauge** of $[a,b]$), we say that $d=(\xi_i,t_i)\in TD_{[a,b]}$ is δ-**fine** if $[t_{i-1},t_i]\subset\big\{t\in[a,b];\ |t-\xi_i|<\delta(\xi_i)\big\}$, for $i=1,2,\cdots,|d|$.

Let us consider functions $f:[a,b]\to X$ and $\alpha:[a,b]\to L(X,Y)$. We say that f is Kurzweil α-integrable (we write $f\in K^\alpha([a,b],X)$) and that $I\in Y$ is its integral $\big($we write $I={}^K\!\int_{[a,b]} d\alpha(s)f(s)\big)$ if given $\varepsilon>0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d=(\xi_i,t_i)\in TD_{[a,b]}$,

$$\Big\|\sum_i \big[\alpha(t_i)-\alpha(t_{i-1})\big]f(\xi_i)-{}^K\!\!\int_{[a,b]} d\alpha(s)f(s)\Big\|<\varepsilon.$$

We denote the indefinite integral of $f \in K^\alpha([a,b],X)$ by $\widetilde{f}^\alpha$ $\big($i.e., $\widetilde{f}^\alpha(t) = {}^K\!\int_{[a,t]} d\alpha(s)\, f(s)$, for every $t \in [a,b]\big)$. We say that f is Henstock α-integrable $\big($we write $f \in H^\alpha([a,b],X)\big)$ if there exists a function $F^\alpha : [a,b] \to Y$ such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_i \big\| \big[\alpha(t_i) - \alpha(t_{i-1})\big] f(\xi_i) - \big[F^\alpha(t_i) - F^\alpha(t_{i-1})\big] \big\| < \varepsilon.$$

In this case we set ${}^H\!\int_{[a,t]} d\alpha(s) f(s) = F^\alpha(t) - F^\alpha(a)$.

Remark 1. *When we consider only constant gauges δ in the definition of $f \in K^\alpha\big([a,b],X\big)$, we obtain the Riemann-Stieltjes integral $\int_{[a,b]} d\alpha(s) f(s)$ and we write $f \in R^\alpha\big([a,b],X\big)$. If $\alpha(t) = t$, then instead of $K^\alpha\big([a,b],X\big)$ and $H^\alpha\big([a,b],X\big)$ we write simply $K\big([a,b],X\big)$ and $H\big([a,b],X\big)$ respectively. We denote by ${}^K\!\int_{[a,b]} f(s)\, ds$ the Kurzweil integral of $f \in K\big([a,b],X\big)$ and by $\widetilde{f}$ its indefinite integral $\big($i.e., $\widetilde{f}(t) = {}^K\!\int_{[a,t]} f(s)\, ds$, for every $t \in [a,b]\big)$. It is immediate that $H^\alpha\big([a,b],X\big) \subset K^\alpha\big([a,b],X\big)$ and, if X is of finite dimension, then $K^\alpha\big([a,b],X\big) = H^\alpha\big([a,b],X\big)$.*

The idea to consider **semi-tagged divisions** $d = (\xi_i, t_i)$ of $[a,b]$ (i.e., (t_i) is a division of $[a,b]$ and $\xi_i \in [a,b]$ for every i, but it is *not* necessary that $\xi_i \in [t_{i-1}, t_i]$ for any i) has originated more restrictive integrals. This idea it is due to E.J. McShane[8], and when it is applied to the definition of the Henstock vector integral, we obtain precisely the Bochner-Lebesgue-Stieltjes integral with finite integral (see [6]). Given functions $f : [a,b] \to X$ and $\alpha : [a,b] \to L(X,Y)$, we write $f \in \mathcal{L}_1^\alpha\big([a,b],X\big)$ if the Bochner-Lebesgue-Stieltjes integral ${}^L\!\int_{[a,b]} d\alpha(s) f(s)$ exists and is finite. Then, the inclusion $\mathcal{L}_1^\alpha\big([a,b],X\big) \subset H^\alpha\big([a,b],X\big)$ holds. If $\alpha(t) = t$, we write simply $\mathcal{L}_1\big([a,b],X\big)$, ${}^L\!\int_{[a,b]} f(s)\, ds$ and $\|f\|_1 = {}^L\!\int_{[a,b]} \|f(s)\|\, ds$ and we have $\mathcal{L}_1\big([a,b],X\big) \subset H\big([a,b],X\big)$. Furthermore, if $f \in H\big([a,b],\mathbb{R}\big)$ is positive, then $f \in \mathcal{L}_1\big([a,b],\mathbb{R}\big)$.

2.2 Basic properties

For a proof of the following result, see [2, Theorem 1.2].

Proposition 1. *Let $\alpha : [a,b] \to L(X,Y)$ and $f \in K^\alpha\big([a,b],X\big)$.*

(i) If $\alpha \in G^\sigma\big([a,b],L(X,Y)\big)$, then $\widetilde{f}^\alpha \in G\big([a,b]Y\big)$.

(ii) If $\alpha \in G^{\sigma-}\big([a,b],L(X,Y)\big)$, then $\widetilde{f}^\alpha \in G^-\big([a,b],Y\big)$.

(iii) If $\alpha \in G^{\sigma+}\big([a,b],L(X,Y)\big)$, then $\widetilde{f}^\alpha \in G^+\big([a,b],Y\big)$.

Remark 2. *It is a consequence of Proposition 1 and its proof that if $\alpha \in C^\sigma\big([a,b],L(X,Y)\big)$ and $f \in K^\alpha\big([a,b],X\big)$, then $\widetilde{f}^\alpha \in C\big([a,b],Y\big)$.*

The next result can be found in [5] or in [2, Theorem 1.5].

Theorem 2. *If either $\alpha \in SV\big([a,b],L(X,Y)\big)$ and $f \in C\big([a,b],X\big)$, or $\alpha \in C\big([a,b],L(X,Y)\big)$ and $f \in BV\big([a,b],X\big)$, then the Riemann-Stieltjes integrals $\int_{[a,b]} d\alpha(s) f(s)$ and $\int_{[a,b]} \alpha(s)\, df(s)$ exist and the integration by parts formula holds:*

$$\int_{[a,b]} d\alpha(s) f(s) = \alpha(b) f(b) - \alpha(a) f(a) - \int_{[a,b]} \alpha(s)\, df(s).$$

In [10, Theorem 15], Schwabik proved the following

Theorem 3. *Let $\alpha \in SV\big([a,b],L(X,Y)\big) \cap G^\sigma\big([a,b],L(X,Y)\big)$ and $f \in G\big([a,b],X\big)$. Then $f \in K^\alpha\big([a,b],X\big)$.*

The reader can find a proof for the next result in [6,9] or in [2, Theorem 1.9].

Theorem 4. *Let $f \in H\big([a,b],X\big)$. Then f is absolutely integrable $\big($i.e., $\|f(\cdot)\| \in \mathcal{L}_1\big([a,b],\mathbb{R}\big)\big)$ if and only if $\widetilde{f} \in BV\big([a,b],X\big)$. In any case, $\|f\|_1 = V(F)$.*

See [2, Theorem 1.10], for a proof of the following

Theorem 5. *If $\alpha \in \mathcal{L}_1\big([a,b],L(X,Y)\big)$ and $f \in G\big([a,b],X\big)$, then $\alpha f \in \mathcal{L}_1\big([a,b],Y\big)$ and ${}^L\!\int_{[a,b]} \alpha(s)f(s)\,ds = {}^K\!\int_{[a,b]} d\widetilde{\alpha}(s)f(s)$.*

Since $C\big([a,b],X\big) \subset G\big([a,b],X\big)$ and $BV\big([a,b],X\big) \subset G\big([a,b],X\big)$ then, in view of Theorem 2, it follows that

Corollary 6. *Suppose $\alpha \in \mathcal{L}_1\big([a,b],L(X,Y)\big)$ and either $f \in C\big([a,b],X\big)$ or $f \in BV\big([a,b],X\big)$. Then $\alpha f \in \mathcal{L}_1\big([a,b],Y\big)$ and ${}^L\!\int_{[a,b]} \alpha(s)f(s)\,ds = \int_{[a,b]} d\widetilde{\alpha}(s)f(s)$.*

Theorem 7. *Let $\alpha \in H\big([a,b],L(X,Y)\big)$ and $f : [a,b] \to X$. Then $\alpha f \in H\big([a,b],Y\big)$ if one of the following conditions is satisfied:*

(i) $f \in BV\big([a,b],X\big)$;

(ii) α is absolutely integrable and $f \in C\big([a,b],X\big)$.

In any case, ${}^K\!\int_{[a,b]} \alpha(s)f(s)\,ds = \int_{[a,b]} d\widetilde{\alpha}(s)f(s)$.

Theorem 7 (i) was proved in [7, Theorem 12.1 and Corollary 12.2]; for a proof of (ii), see Theorem 4 above and [2, Theorem 1.8].

If $\alpha \in SV\big([a,b],L(X)\big) \cap G^{\sigma-}\big([a,b],L(X)\big)$ and $f \in G^-\big([a,b],X\big)$, then the Kurzweil vector integral ${}^K\!\int_{[a,t]} d\alpha(s)f(s)$ exists for every $t \in [a,b]$ (by Theorem 3) and $\big\|{}^K\!\int_{[a,t]} d\alpha(s)f(s)\big\| \leq SV(\alpha)\ \|f\|_\infty$ (see [10, Proposition 10]). Thus, by Proposition 1, we can define an operator $F_\alpha : G^-\big([a,b],X\big) \to G^-\big([a,b],X\big)$ by $F_\alpha f(t) = {}^K\!\int_{[a,t]} d\alpha(s)f(s)$, for every $t \in [a,b]$. Then the next result can be easily proved.

Proposition 8. *Suppose that $\alpha \in SV\big([a,b],L(X)\big) \cap G^{\sigma-}\big([a,b],L(X)\big)$, $f \in G^-([a,b],X)$ and $F_\alpha f(t) = {}^K\!\int_{[a,t]} d\alpha(s)f(s)$, for every $t \in [a,b]$. Then $F_\alpha \in L\big(G^-([a,b],X)\big)$.*

In an analogous way, applying Theorem 2 one can show that

Proposition 9. *If $\alpha \in SV\big([a,b],L(X)\big) \cap C^\sigma\big([a,b],L(X)\big)$, $f \in C\big([a,b],X\big)$ and $F_\alpha f(t) = \int_{[a,t]} d\alpha(s)f(s)$, for every $t \in [a,b]$, then $F_\alpha \in L\big(C([a,b],X)\big)$.*

Proposition 10. *Suppose that $\alpha \in H\big([a,b],L(X)\big)$ is absolutely integrable, $f \in BV_a^-\big([a,b],X\big)$ and $F_{\widetilde{\alpha}} f(t) = \int_{[a,t]} d\widetilde{\alpha}(s)f(s)$, for every $t \in [a,b]$. Then $F_{\widetilde{\alpha}} \in L\big(BV_a^-\big([a,b],X\big)\big)$.*

Proof. We will prove that $F_{\widetilde{\alpha}} f \in BV_a^-\big([a,b],X\big)$, for $f \in BV_a^-\big([a,b],X\big)$. The linearity and continuity of $F_{\widetilde{\alpha}}$ will be left to the reader. From the remark after Proposition 1, $\widetilde{\alpha} \in C\big([a,b],L(X)\big)$ and $F_{\widetilde{\alpha}} f \in C\big([a,b],X\big)$. Besides, $F_{\widetilde{\alpha}} f(a) = 0$. Therefore, it is sufficient to show that $F_{\widetilde{\alpha}} f \in BV\big([a,b],X\big)$.

We assert that $f \in H^{\widetilde{\alpha}}\big([a,b],X\big)$. Indeed, since $\alpha \in H\big([a,b],L(X)\big)$ and $f \in BV\big([a,b],X\big)$, it follows by Theorem 7 that $\alpha f \in H\big([a,b],X\big)$ with ${}^K\!\int_{[a,b]} \alpha(s)f(s)\,ds = \int_{[a,b]} d\widetilde{\alpha}(s)f(s)$. Let $\varepsilon > 0$ and δ be a gauge of $[a,b]$ such that for every δ -fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_i \Big\| {}^K\!\!\int_{[t_{i-1},t_i]} \alpha(s)\,ds - \alpha(\xi_i)[t_i - t_{i-1}] \Big\| < \varepsilon$$

and

$$\sum_i \Big\| {}^K\!\!\int_{[t_{i-1},t_i]} \alpha(s)f(s)\,ds - \alpha(\xi_i)f(\xi_i)[t_i - t_{i-1}] \Big\| < \varepsilon.$$

Then,

$$\begin{aligned}&\sum_i \Big\| {}^K\!\!\int_{[t_{i-1},t_i]} d\widetilde{\alpha}(s)f(s) - \big[\widetilde{\alpha}(t_i) - \widetilde{\alpha}(t_{i-1})\big]f(\xi_i) \Big\| \\ &= \sum_i \Big\| {}^K\!\!\int_{[t_{i-1},t_i]} \alpha(s)f(s)\,ds - \big[\widetilde{\alpha}(t_i) - \widetilde{\alpha}(t_{i-1})\big]f(\xi_i) \Big\|\end{aligned}$$

$$\leq \sum_i \left\| {}^K\!\!\int_{[t_{i-1},t_i]} \alpha(s)f(s)\,ds - \alpha(\xi_i)f(\xi_i)[t_i - t_{i-1}] \right\|$$
$$+ \sum_i \left\| \alpha(\xi_i)f(\xi_i)[t_i - t_{i-1}] - \big[\widetilde{\alpha}(t_i) - \widetilde{\alpha}(t_{i-1})\big] f(\xi_i) \right\|$$
$$< \varepsilon + \sum_i \left\| \alpha(\xi_i)[t_i - t_{i-1}] - {}^K\!\!\int_{[t_{i-1},t_i]} \alpha(s)\,ds \right\| \|f\|_\infty < \varepsilon + \varepsilon \|f\|_\infty$$

and $f \in H^{\widetilde{\alpha}}\big([a,b],X\big)$.

Now, given $\varepsilon > 0$ and the gauge δ of $[a,b]$ from the definition of $f \in H^{\widetilde{\alpha}}\big([a,b],X\big)$, let $d = (\xi_i, t_i) \in TD_{[a,b]}$ be δ-fine. Hence,

$$\sum_i \left\| F_{\widetilde{\alpha}} f(t_i) - F_{\widetilde{\alpha}} f(t_{i-1}) \right\| = \sum_i \left\| {}^K\!\!\int_{[t_{i-1},t_i]} d\widetilde{\alpha}(s)f(s) \right\|$$
$$\leq \sum_i \left\| {}^K\!\!\int_{[t_{i-1},t_i]} d\widetilde{\alpha}(s)f(s) - \big[\widetilde{\alpha}(t_i) - \widetilde{\alpha}(t_{i-1})\big] f(\xi_i) \right\|$$
$$+ \sum_i \left\| \big[\widetilde{\alpha}(t_i) - \widetilde{\alpha}(t_{i-1})\big] f(\xi_i) \right\| < \varepsilon + V(\widetilde{\alpha}) \|f\|_\infty,$$

which implies that $F_{\widetilde{\alpha}} f \in BV\big([a,b],X\big)$, once $V(\widetilde{\alpha}) < \infty$ (Theorem 4).

3 Linear Volterra-Stieltjes Integral Equations

The nest result can be found in [1, Theorem 3.2].

Theorem 11 (Barbanti). *Given $\alpha \in SV\big([a,b],L(X)\big) \cap G^{\sigma-}\big([a,b],L(X)\big)$, consider the following linear integral equation of Volterra-Kurzweil-Stieltjes*

$$x(t) + {}^K\!\!\int_{[a,t]} d\alpha(s)x(s) = f(t), \qquad t \in [a,b], \tag{5}$$

where $x, f \in G^-\big([a,b],X\big)$. If $(I + F_\alpha) \in L\big(G^-([a,b],X)\big)$ is a Fredholm operator, where $F_\alpha f(t) = {}^K\!\int_{[a,t]} d\alpha(s)f(s)$, $t \in [a,b]$, and I is the identity operator, then given $f \in G^-\big([a,b],X\big)$, there exists $x \in G^-\big([a,b],X\big)$ satisfying equation (5).

Owing to Proposition 9, following the ideas of Barbanti[1], one can prove Theorem 12 below.

Theorem 12. *Given $\alpha \in SV\big([a,b],L(X)\big) \cap C^\sigma\big([a,b],L(X)\big)$, consider the following linear integral equation of Volterra-Stieltjes*

$$x(t) + \int_{[a,t]} d\alpha(s)x(s) = f(t), \qquad t \in [a,b], \tag{6}$$

where $x,\ f \in C\big([a,b],X\big)$. If $(I + F_\alpha) \in L\big(C([a,b],X)\big)$ is a Fredholm operator, where $F_\alpha f(t) = \int_{[a,t]} d\alpha(s)f(s)$, for every $t \in [a,b]$, and I is the identity operator, then given $f \in C\big([a,b],X\big)$, there exists $x \in C\big([a,b],X\big)$ satisfying equation (6).

Theorem 13. *Given $\alpha \in H\big([a,b],L(X)\big)$ absolutely integrable, consider the following linear integral equation of Volterra-Stieltjes*

$$x(t) + \int_{[a,t]} d\widetilde{\alpha}(s)x(s) = f(t), \qquad t \in [a,b], \tag{7}$$

where $x,\ f \in BV_a^-\big([a,b],X\big)$. If $(I+F_{\widetilde{\alpha}}) \in L\big(BV_a^-([a,b],X)\big)$ is a Fredholm operator, where $F_{\widetilde{\alpha}}f(t) = \int_{[a,t]} d\widetilde{\alpha}(s)f(s)$, for every $t \in [a,b]$, and I is the identity operator, then given $f \in BV_a^-\big([a,b],X\big)$, there exists $x \in BV_a^-\big([a,b],X\big)$ satisfying equation (7).

Proof. By the Remark after Proposition 1 and by Theorem 4, $\widetilde{\alpha} \in BV\big([a,b],L(X)\big) \cap C^\sigma\big([a,b], L(X)\big)$. Then by Proposition 10, $(I+F_{\widetilde{\alpha}}) \in L\big(BV_a^-([a,b],X)\big)$. The rest of the proof follows the steps of [1, Theorem 2.3] noticing that $BV\big([a,b],L(X)\big) \subset SV\big([a,b],L(X)\big)$, $BV_a^-\big([a,b],X\big) \subset G_a([a,b],X)$ and $BV_a^-\big([a,b],X\big)$ is a Banach space.

Remark 3. *The important aspect of Theorems 11–13 is that each integral equation can be seen as the limit of discrete systems which means that one can study the properties of such equations through the transfer of the properties of the corresponding discrete systems (see [1]).*

4 Linear Volterra Integral Equations

The results of the present section are part of [3].

4.1 Linear Volterra-Bochner-Lebesgue Integral Equations

We consider the next linear integral equation of Volterra in the sense of the Bochner-Lebesgue integral

$$x(t) + {}^{L}\!\!\int_{[a,t]} \alpha(s)x(s)\,ds = f(t), \qquad t \in [a,b], \tag{8}$$

in the following cases:

(a) $\alpha \in \mathcal{L}_\infty\big([a,b],L(X)\big)$ and $x, f \in G^-\big([a,b],X\big)$;

(b) $\alpha \in \mathcal{L}_1\big([a,b],L(X)\big)$ and $x,\ f \in C\big([a,b],X\big)$;

(c) $\alpha \in \mathcal{L}_1\big([a,b],L(X)\big)$ and $x,\ f \in BV_a^-\big([a,b],X\big)$.

In each case, it follows from either Theorem 5 or its Corollary that equation (8) is equivalent to the following equation

$$x(t) + {}^{*}\!\!\int_{[a,t]} d\widetilde{\alpha}(s)x(s) = f(t), \qquad t \in [a,b], \tag{9}$$

where ${}^*\!\int$ denotes either the Kurzweil or the Riemann integral and $\widetilde{\alpha}$ is the indefinite integral of α. We obtain the results for equation (8) by applying one of the results from the previous section to equation (9).

Theorem 14. *Let ${}^*\!\int$ denote either the Kurzweil or the Riemann integral and let $I\big([a,b],X\big)$ denote one of the spaces $G^-\big([a,b],X\big)$, $C\big([a,b],X\big)$ or $BV_a^-\big([a,b],X\big)$. Given $\alpha \in \mathcal{L}_1\big([a,b],L(X)\big)$, consider equation (8), where $x,\ f \in I\big([a,b],X\big)$. If $(I+F_{\widetilde{\alpha}}) \in L\big(I([a,b],X)\big)$ is a Fredholm operator, where $F_{\widetilde{\alpha}}f(t) = {}^*\!\int_{[a,t]} d\widetilde{\alpha}(s)f(s)$, $t \in [a,b]$, and I is the identity operator, then given $f \in I\big([a,b],X\big)$, there exists $x \in I\big([a,b],X\big)$ satisfying (8).*

Proof. We prove the case when $I\big([a,b],X\big) = G^-\big([a,b],X\big)$. By Theorem 5, (8) is equivalent to the Volterra-Kurzweil-Stieltjes linear integral equation (5). By the remark after Proposition 1, $\widetilde{\alpha} \in C\big([a,b],L(X)\big)$. Then, since $\mathcal{L}_1\big([a,b],L(X)\big) \subset H\big([a,b],L(X)\big)$, $\widetilde{\alpha} \in BV\big([a,b],L(X)\big)$ (Theorem 4), and the result follows by Theorem 11. The other cases follow in an analogous way using the Corollary after Theorem 5 and one of the Theorems 12 or 13.

4.2 Linear Volterra-Henstock Integral Equations

We now consider the linear integral equation of Volterra in the sense of the Henstock integral

$$x(t) + {}^{K}\!\!\int_{[a,t]} \alpha(s)x(s)\,ds = f(t), \qquad t \in [a,b], \tag{10}$$

in the following cases:

(a) $\alpha \in H\big([a,b], L(X)\big)$ is bounded and $x,\ f \in BV_a^-\big([a,b], X\big)$;

(b) $\alpha \in H\big([a,b], L(X)\big)$ is absolutely integrable and $x,\ f \in C([a,b], X)$.

As we did in [2], we establish the results for equation (10) through the analysis of a sequence of equations of type (8). And this is done by means of

Lemma 15[4]**.** *If $f \in H\big([a,b], X\big)$, then there exists a sequence of closed sets $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \uparrow [a,b]$ (i.e., $X_n \subset X_{n+1} \subset [a,b]$ for every $n \in \mathbb{N}$, and $\cup X_n = [a,b]$) and $f \in \mathcal{L}_1(X_n, X)$, for every $n \in \mathbb{N}$. Furthermore,*

$$\lim_{n\to\infty} {}^{L}\!\!\int_{X_n \cap [a,t]} f(s)\, ds = {}^{K}\!\!\int_{[a,t]} f(s)\, ds$$

uniformly for every $t \in [a,b]$.

The above result was proved originally for $X = \mathbb{R}$. But with obvious adaptations, it also holds for the case when X is a Banach space.

Theorem 16. *Suppose that $\alpha \in H\big([a,b], L(X)\big)$ is bounded (respectively $\alpha \in H\big([a,b], L(X)\big)$ is absolutely integrable). Consider equation (10), the linear integral equations of Volterra-Bochner-Lebesgue obtained through Lemma 15*

$$x(t) + {}^{L}\!\!\int_{[a,t]} \chi_{X_n}(s)\alpha(s)x(s)\, ds = f(t), \qquad t \in [a,b], \quad n \in \mathbb{N}, \tag{11}$$

and the operator $T : BV_a^-\big([a,b], X\big) \to BV_a^-\big([a,b], X\big)$ $\big($respectively $T : C\big([a,b], X\big) \to C\big([a,b], X\big)\big)$ defined by

$$(Tx)(t) = f(t) - {}^{K}\!\!\int_{[a,t]} \alpha(s)x(s)\, ds, \qquad t \in [a,b],$$

where $x,\ f \in BV_a^-\big([a,b], X\big)$$\big($resp. $x,\ f \in C\big([a,b], X\big)\big)$. If $(I + F_{\widetilde{\alpha}_n}) \in L\big(BV_a^-([a,b], X)\big)$ $\big($resp. $(I + F_{\widetilde{\alpha}_n}) \in L\big(C([a,b], X)\big)\big)$ is a Fredholm operator, where $\widetilde{\alpha}_n(t) = {}^{L}\!\int_{[a,t]} \chi_{X_n}(s)\alpha(s)\, ds$ and $F_{\widetilde{\alpha}_n} f(t) = \int_{[a,t]} d\widetilde{\alpha}_n(s) f(s)$, $t \in [a,b]$, and I is the identity operator, then for every $f \in BV_a^-\big([a,b], X\big)$ $\big($resp. $f \in C\big([a,b], X\big)\big)$ and $n \in \mathbb{N}$, equation (11) admits a solution $x_n \in BV_a^-\big([a,b], X\big)$ $\big($resp. $x_n \in C\big([a,b], X\big)\big)$. Suppose in addition that one of the following conditions is satisfied:

(i) $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x_0 \in BV_a^-\big([a,b], X\big)$ $\big($resp. $x_0 \in C\big([a,b], X\big)\big)$;

(ii) α is bounded and T^m is a contraction for some $m > 1$, where T^m is the composition of T m times.

If (i) holds, then x_0 is a solution of (10). If (ii) holds, then there exists $x = \lim_n x_n$, $x \in BV_a^-\big([a,b], X\big)$ $\big($resp. $x \in C\big([a,b], X\big)\big)$, such that (10) is fulfilled.

Proof. For each $n \in \mathbb{N}$, we consider the continuous mapping $T_n : BV_a^-\big([a,b], X\big) \to BV_a^-\big([a,b], X\big)$ $\big($resp. $T_n : C\big([a,b], X\big) \to C\big([a,b], X\big)\big)$ given by

$$(T_n x)(t) = f(t) - {}^{L}\!\!\int_{[a,t]} \chi_{X_n}(s)\alpha(s)x(s)\, ds, \qquad t \in [a,b].$$

By Theorem 14, each equation (11) admits a solution $x_n \in BV_a^-\big([a,b], X\big)$ $\big($resp. $x_n \in C\big([a,b], X\big)\big)$. The rest of the demonstration follows the steps of [2, Theorem 2.4] which uses Lemma 15 and a Fixed Point Theorem for sequences of mappings.

When we take $\alpha \in \mathcal{L}_1\big([a,b], L(X)\big)$ in Theorem 16, we use the fact that $\|\alpha\|_1 < \infty$ instead of α bounded (i.e., $\|\alpha\|_\infty < \infty$). In this case, $\widetilde{\alpha} \in BV\big([a,b], L(X)\big)$ (see Theorem 4) and there is a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ such that each X_n is the finite union of closed nonoverlapping intervals, $X_n \uparrow [a,b]$, and $\alpha \in \mathcal{L}_1\big(X_n, L(X)\big)$, for every $n \in \mathbb{N}$ (see [2], the comments after Theorem 3.1). Under these considerations we have

Theorem 17. *Let $\alpha \in L_1\big([a,b], L(X)\big)$ and $I\big([a,b],X\big)$ denote one of the spaces $C\big([a,b],X\big)$ or $BV_a^-\big([a,b],X\big)$. Consider the linear integral equations of Volterra-Bochner-Lebesgue (8) and (11) and $T : I\big([a,b],X\big) \to I\big([a,b],X\big)$ given by*

$$(Tx)(t) = f(t) - {}^L\!\!\int_{[a,t]} \alpha(s)x(s)\,ds, \qquad t \in [a,b],$$

where $x, f \in I\big([a,b],X\big)$ and $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of sets as in the previous paragraph. If $(I + F_{\widetilde{\alpha}_n}) \in L\big(I([a,b],X)\big)$ is a Fredholm operator, where $\widetilde{\alpha}_n(t) = {}^L\!\int_{[a,t]} \chi_{X_n}(s)\alpha(s)\,ds$ and $F_{\widetilde{\alpha}_n} f(t) = \int_{[a,t]} d\widetilde{\alpha}_n(s) f(s)$, $t \in [a,b]$, and I is the identity operator, then given $n \in \mathbb{N}$ and $f \in I\big([a,b],X\big)$, equation (11) admits a solution $x_n \in I\big([a,b],X\big)$. Suppose in addition that one of the following conditions is satisfied:

(i) $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x_0 \in I\big([a,b],X\big)$;

(ii) T^m is a contraction for some $m > 1$, where T^m is the composition of T m times.

If (i) holds, then x_0 is a solution of (8). If (ii) holds, then there exists $x = \lim_n x_n \in I\big([a,b],X\big)$ satisfying (8).

The proof of Theorem 17 is analogous to the proof of Theorem 16, replacing the integral of Henstock by the Bochner-Lebesgue's.

Remark 4. *When $X = \mathbb{R}$, then $\alpha \in H\big([a,b], L(\mathbb{R})\big)$ absolutely integrable belongs to $\mathcal{L}_1\big([a,b], L(\mathbb{R})\big)$. Hence equation (10) with x, $f \in C\big([a,b],X\big)$ coincides with equation (8).*

Remark 5. *If, for instance, the composition of $(F_{\widetilde{\alpha}_n})$ k times, $(F_{\widetilde{\alpha}_n})^k$, is a compact operator for some positive integer k, then $(I + F_{\widetilde{\alpha}_n})$ (from either Theorem 16 or Theorem 17) is a compact operator (see [9]).*

References

[1] Barbanti, L. Linear Stieltjes equation with generalized Riemann integral and existence of regulated solutions. *Acta Math. Appl. Sinica*, 17(4): 526–531 (2001)

[2] Federson, M., Bianconi, R. Linear integral equations of Volterra concerning the integral of Henstock. *Real Anal. Exchange*, 25(1): 389–417 (2000)

[3] Federson, M. Sobre a existência de soluções para Equações integrais Lineares com respeito a integrais de Gauge. Doctor Thesis, Institute of Mathematics and Statistics, University of São Paulo, Brazil (1998)

[4] Gengian, L. On necessary conditions for Henstock integrability. *Real Anal. Exchange*, 18: 522–531 (1992–93)

[5] Hönig, C.S. Volterra-Stieltjes integral equations. In: Math. Studies, Vol.16, ed. by North-Holland Publ. Comp., Amsterdam (1975)

[6] Hönig, C.S. A Riemaniann characterization of the Bochner-Lebesgue integral. *Seminário Brasileiro de Análise*, 35: 351–358 (1992)

[7] Lee, P.Y. Lanzhou lectures on Henstock integration. World Sci., Singapore (1989)

[8] McShane, E.J. A unified theory of integration. *Am. Math. Monthly*, 80: 349–359 (1973)

[9] Schechter, M. Principles of functional analysis. Academic Press, New York (1971)

[10] Schwabik, S. Abstract Perron-Stieltjes integral. *Math. Bohem.*, 121(4): 425–447 (1996)