SUBSTITUTION FORMULAS FOR THE KURZWEIL AND HENSTOCK VECTOR INTEGRALS

M. FEDERSON, Brazil

Abstract

Results on integration by parts and integration by substitution for the variational integral of Henstock are well-known ([3], [7]).When real-valued functions are considered, such results also hold for the Generalized Riemann Integral defined by Kurzweil since, in this case, the integrals of Kurzweil and Henstock coincide. However, in a Banach-space valued context, the Kurzweil integral properly contains that of Henstock ([2]). In the present paper, we consider abstract vector integrals of Kurzweil and prove Substitution Formulas by functional analytic methods. In general, Substitution Formulas may not hold for Kurzweil vector integrals even if they are defined ([1]).

Keywords: Kurzweil-Henstock integrals, integration by parts, integration by substitution.
Mathematical Subject Classification (1991): 26A39.

1 Preliminaries

Throughout this paper, $[a,b]$ is a compact interval of the real line $\mathbb{R}$, X and Y are Banach spaces and $L(X,Y)$ is the Banach space of all linear continuous functions from X to Y.

Let $C([a,b],X)$ and $G([a,b],X)$ be respectively the Banach spaces of continuous and of regulated functions from $[a,b]$ to X endowed with the supremum norm which we denote by $\|\cdot\|_\infty$. We recall that a function $f:[a,b]\to X$ is regulated if it has only discontinuities of the first kind, i.e. if the onesided limits $\lim_{t\to c+} f(t)$ and $\lim_{t\to c-} f(t)$ exist for every $c\in[a,b]$. Let $C^\sigma([a,b],L(X,Y))$ be the set of all functions $\alpha:[a,b]\to L(X,Y)$ which are weakly continuous (i.e. the function $t\in[a,b]\to\alpha(t)x\in Y$ is continuous for every $x\in X$) and let $G^\sigma([a,b],L(X,Y))$ be the set of all weakly regulated functions $\alpha:[a,b]\to L(X,Y)$ (i.e. the function $t\in[a,b]\to\alpha(t)x\in Y$ is regulated for every $x\in X$). See [4].

Let $a=t_0<t_1<\ldots<t_n=b$ be a division of $[a,b]$. In this case we write $d=(t_i)\in D_{[a,b]}$ and $|d|=n$. Given $d=(t_i)\in D_{[a,b]}$ and functions $\alpha:[a,b]\to L(X,Y)$ and

$f : [a, b] \to X$, we define

$$V_d(f) = \sum_i \|f(t_i) - f(t_{i-1})\|$$

and

$$SV_d(\alpha) = \sup\left\{\left\|\sum_i [\alpha(t_i) - \alpha(t_{i-1})] y_i\right\|; y_i \in Y, \|y_i\| \le 1\right\}.$$

Then

$$V(f) = \sup\left\{V_d(f); d \in D_{[a,b]}\right\}$$

is the variation of f and

$$SV(\alpha) = \sup\left\{SV_d(\alpha); d \in D_{[a,b]}\right\}$$

is the semivariation of α. If $V(f) < \infty$, then f is of bounded variation and we write $f \in BV([a, b], X)$. If $SV(\alpha) < \infty$, then α is of bounded semivariation and we write $\alpha \in SV([a, b], L(X, Y))$. Clearly $BV([a, b], L(X, Y)) \subset SV([a, b], L(X, Y))$. Let $X' = L(X, \mathbb{R})$. Then $SV([a, b], L(X, \mathbb{R})) = BV([a, b], X')$. Moreover, for $L(X) = L(X, X)$, $SV([a, b], L(X)) = BV([a, b], L(X))$ if and only if X is of finite dimension.

For more information about these spaces, see [4].

2 Basic definitions and properties

When $d = (t_i) \in D_{[a,b]}$ and $\xi_i \in [t_{i-1}, t_i]$, for $i = 1, 2, \ldots, |d|$, then $d = (\xi_i, t_i)$ is called a tagged division of $[a, b]$. We denote by $TD_{[a,b]}$ the set of all tagged divisions of $[a, b]$. A gauge of $[a, b]$ is a function $\delta : [a, b] \to]0, \infty[$ and $d = (\xi_i, t_i) \in TD_{[a,b]}$ is called δ-fine if $[t_{i-1}, t_i] \subset \{t \in [a, b]; |t - \xi_i| < \delta(\xi_i)\}$, for $i = 1, 2, \ldots, |d|$.

In what follows we consider functions $f : [a, b] \to X$ and $\alpha : [a, b] \to L(X, Y)$.

We say that f is Kurzweil α-integrable (we write $f \in K^\alpha([a, b], X)$) and that $I \in Y$ is its integral (we write $I = {}^K\!\int_{[a,b]} d\alpha(t)\, f(t)$) if given $\varepsilon > 0$, there is a gauge δ of $[a, b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\|\sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - {}^K\!\int_{[a,b]} d\alpha(t)\, f(t)\right\| < \varepsilon.$$

In particular, if we consider only constant gauges, then we obtain the Riemann-Stieltjes integral $\int_{[a,b]} d\alpha(t)\, f(t)$. We denote by $R^\alpha([a, b], X)$ the set of all $f : [a, b] \to X$ for which the Riemann-Stieltjes integral with respect to α exists.

Analogously we say that α is Kurzweil f-integrable (we write $\alpha \in K_f([a,b], L(X,Y))$) and that $I \in Y$ is its integral (we write $I = {}^K\int_{[a,b]} \alpha(t)\, df(t)$) if for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\| \sum_{i=1}^{|d|} \alpha(\xi_i)[f(t_i) - f(t_{i-1})] - {}^K\int_{[a,b]} \alpha(t)\, df(t) \right\| < \varepsilon.$$

Again, when constant gauges are considered, we obtain the space of Riemann-Stieltjes integrals $\int_{[a,b]} \alpha(t)\, df(t)$ denoted by $R_f([a,b], L(X,Y))$.

Let $f \in K^\alpha([a,b], X)$. Its indefinite integral $\tilde{f}^\alpha : [a,b] \to Y$ is given by $\tilde{f}^\alpha(t) = {}^K\int_{[a,t]} d\alpha(s)\, f(s)$, for every $t \in [a,b]$. Similarly, we denote the indefinite integral of $\alpha \in K_f([a,b], L(X,Y))$ by $\tilde{\alpha}_f : [a,b] \to Y$, that is, $\tilde{\alpha}_f(t) = {}^K\int_{[a,t]} \alpha(s)\, df(s)$, for every $t \in [a,b]$. When $\alpha(t) = t$, we write simply $K([a,b], X)$, $R([a,b], X)$ and $\tilde{f}$ instead of $K^\alpha([a,b], X)$, $R^\alpha([a,b], X)$ and $\tilde{f}^\alpha$ respectively.

We say that f is Henstock α-integrable (we write $f \in H^\alpha([a,b], X)$) if there is a function $F^\alpha : [a,b] \to Y$ such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_{i=1}^{|d|} \left\| [\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - [F^\alpha(t_i) - F^\alpha(t_{i-1})] \right\| < \varepsilon.$$

We set ${}^H\int_{[a,b]} d\alpha(t)\, f(t) = F^\alpha(b) - F^\alpha(a)$ in this case. Analogously, α is Henstock f-integrable (we write $\alpha \in H_f([a,b], L(X,Y))$) if there is a function $A_f : [a,b] \to Y$ such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_{i=1}^{|d|} \left\| \alpha(\xi_i)[f(t_i) - f(t_{i-1})] - [A_f(t_i) - A_f(t_{i-1})] \right\| < \varepsilon.$$

In this case we set ${}^H\int_{[a,b]} \alpha(t)\, df(t) = A_f(b) - A_f(a)$.

The inclusions

$$H^\alpha([a,b], X) \subset K^\alpha([a,b], X) \quad \text{and} \quad H_f([a,b], L(X,Y)) \subset K_f([a,b], L(X,Y))$$

hold. Besides, if $f \in H^\alpha([a,b], X)$, then $F^\alpha(t) - F^\alpha(a) = {}^K\int_{[a,t]} d\alpha(t)\, f(t) = \tilde{f}^\alpha(t) - \tilde{f}^\alpha(a)$, for every $t \in [a,b]$. Analogously given $\alpha \in H_f([a,b], L(X,Y))$, $A_f(t) - A_f(a) = {}^K\int_{[a,t]} \alpha(t)\, df(t) = \tilde{\alpha}_f(t) - \tilde{\alpha}_f(a)$, for every $t \in [a,b]$. Similarly as above if $\alpha(t) = t$, we write $H([a,b], X)$ instead of $H^\alpha([a,b], X)$.

See [2], Theorem 1.2 for a proof of the next result.

Theorem 2.1 *If* $\alpha \in G^{\sigma}([a,b], L(X,Y))$ *(respectively* $\alpha \in C^{\sigma}([a,b], L(X,Y))$*) and* $f \in K^{\alpha}([a,b], X)$*, then* $\tilde{f}^{\alpha} \in G([a,b], X)$ *(respectively* $\tilde{f}^{\alpha} \in C([a,b], X)$*). Analogously, if* $f \in G([a,b], X)$ *(respectively* $f \in C([a,b], X)$*) and* $\alpha \in K_f([a,b], L(X,Y))$*, then* $\tilde{\alpha}_f \in G([a,b], Y)$ *(respectively* $\tilde{\alpha}_f \in C([a,b], Y)$*).*

A proof of the next result can be found in [2], Theorem 1.5.

Theorem 2.2 *Suppose either* $\alpha \in SV([a,b], L(X,Y))$ *and* $f \in C([a,b], X)$*, or* $\alpha \in C([a,b], L(X,Y))$ *and* $f \in BV([a,b], X)$*. Then the Riemann-Stieltjes integrals*

$$\int_{[a,b]} d\alpha(t)\ f(t) \quad \text{and} \quad \int_{[a,b]} \alpha(t)\ df(t)$$

exist and the Integration by Parts Formula holds:

$$\int_{[a,b]} d\alpha(t)\ f(t) = \alpha(b) f(b) - \alpha(a) f(a) - \int_{[a,b]} \alpha(t)\ df(t)\ .$$

A proof of the next theorem can be found in [9], Theorem 15.

Theorem 2.3 *If* $f \in G([a,b], X)$*,* $\alpha \in SV([a,b], L(X,Y))$ *and* $\alpha \in G^{\sigma}([a,b], L(X,Y))$*, then* $f \in K^{\alpha}([a,b], X)$*.*

With a proof analogous to that of Schwabik for the Theorem 2.3 one can show that

Theorem 2.4 *Suppose* $f \in BV([a,b], X)$ *and* $\alpha \in G([a,b], L(X,Y))$*. Then* $\alpha \in K_f([a,b], L(X,Y))$*.*

A proof of the next result can be found in [9], Lemma 16.

Lemma 2.1 (Saks-Henstock) *The following assertions hold.*

(i) Let $\alpha : [a,b] \to L(X,Y)$ *and* $f \in K^{\alpha}([a,b], X)$*. Given* $\varepsilon > 0$*, let the gauge* δ *of* $[a,b]$ *be such that for every* δ*-fine* $d = (\xi_i, t_i) \in TD_{[a,b]}$*,*

$$\left\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - {}^{K}\!\!\int_{[a,b]} d\alpha(t)\ f(t) \right\| < \varepsilon.$$

Then for $a \le c_1 \le \eta_1 \le d_1 \le c_2 \le \eta_2 \le d_2 \le \ldots \le c_k \le \eta_k \le d_k \le b$ *with* $[c_j, d_j] \subset]\eta_j - \delta(\eta_j), \eta_j + \delta(\eta_j)[$ *for every* j*,*

$$\left\| \sum_{j=1}^{k} \left\{ [\alpha(d_j) - \alpha(c_j)] f(\eta_j) - {}^{K}\!\!\int_{[c_j,d_j]} d\alpha(t)\ f(t) \right\} \right\| \le \varepsilon.$$

(ii) Let $f:[a,b]\to X$ *and* $\alpha\in K_f([a,b],L(X,Y)).$ *If for* $\varepsilon>0$, *the gauge* δ *of* $[a,b]$ *is such that for every* δ*-fine* $d=(\xi_i,t_i)\in TD_{[a,b]}$,

$$\left\|\sum_{i=1}^{|d|}\alpha(\xi_i)\left[f(t_i)-f(t_{i-1})\right]-{}^K\!\int_{[a,b]}\alpha(t)\,df(t)\right\|<\varepsilon,$$

then for $a\le c_1\le\eta_1\le d_1\le c_2\le\eta_2\le d_2\le\ldots\le c_k\le\eta_k\le d_k\le b$ *with* $[c_j,d_j]\subset\,]\eta_j-\delta(\eta_j),\eta_j+\delta(\eta_j)[$ *for every* j,

$$\left\|\sum_{j=1}^{k}\left\{\alpha(\eta_j)\left[f(d_j)-f(c_j)\right]-{}^K\!\int_{[c_j,d_j]}\alpha(t)\,df(t)\right\}\right\|\le\varepsilon.$$

3 Integration by Parts Formulas

Using the definitions it is not difficult to show that the following holds.

Theorem 3.1 *Suppose* $f\in H([a,b],X)$ *and* $\alpha\in K_{\tilde f}([a,b],L(X,Y))$ *(respectively* $\alpha\in H_{\tilde f}([a,b],L(X,Y))$*) is bounded. Then* $\alpha f\in K([a,b],Y)$ *(respectively* $\alpha f\in H([a,b],Y)$*) and*

$${}^K\!\int_{[a,b]}\alpha(t)\,f(t)\,dt={}^K\!\int_{[a,b]}\alpha(t)\;d\tilde f(t).$$

Corollary 3.1 *Let* $\alpha\in G([a,b],L(X,Y))$ *and* $f\in H([a,b],X)$ *be such that* $\tilde f\in BV([a,b],X)$. *Then* $\alpha f\in K([a,b],Y)$ *and*

$${}^K\!\int_{[a,b]}\alpha(t)\,f(t)\,dt={}^K\!\int_{[a,b]}\alpha(t)\;d\tilde f(t).$$

P r o o f. By Theorem 2.4, the Kurzweil vector integral ${}^K\!\int_{[a,b]}\alpha(t)\,d\tilde f(t)$ exists. □

In the next corollaries, we use the fact that the Riemann-Stieltjes integrals are special cases of the Kurzweil vector integrals.

Corollary 3.2 *Let* $\alpha\in C([a,b],L(X,Y))$ *and* $f\in H([a,b],X)$ *be such that* $\tilde f\in BV([a,b],X)$. *Then* $\alpha f\in K([a,b],Y)$ *with*

$${}^K\!\int_{[a,b]}\alpha(t)\,f(t)\,dt=\int_{[a,b]}\alpha(t)\;d\tilde f(t)\tag{1}$$

and

$$\int_{[a,b]}\alpha(t)\;d\tilde f(t)=\tilde\alpha(b)\,f(b)-\tilde\alpha(a)\,f(a)-\int_{[a,b]}d\alpha(t)\;\tilde f(t).\tag{2}$$

P r o o f. Follows by Theorem 2.2. □

Corollary 3.3 *If* $f \in H([a,b],X)$ *and* $\alpha \in SV([a,b],L(X,Y))$*, then* $\alpha f \in K([a,b],Y)$ *and equations (1) and (2) hold.*

P r o o f. Follows by Theorems 2.1 and 2.2. □

It is even true that

Theorem 3.2 *If* $f \in K([a,b],X)$ *and* $\alpha \in SV([a,b],L(X,Y))$*, then* $\alpha f \in K([a,b],Y)$ *and equations (1) and (2) hold.*

The proof of the theorem above is due to Hnig (see [5]). It follows the steps of the proof of Theorem 4.1 below applying Theorems 2.1 and 2.2. When the functions are real-valued, $SV([a,b],L(\mathbb{R})) = BV([a,b],L(\mathbb{R}))$ and Theorem 3.2 is also proved in [7, Theorem 12.1 and Corollary 12.2].

On the other hand,

Theorem 3.3 *If* $f \in H([a,b],X)$ *and* $\alpha \in BV([a,b],L(X,Y))$*, then* $\alpha f \in H([a,b],Y)$ *and equations (1) and (2) hold.*

P r o o f. By hypothesis $f \in H([a,b],X)$. By Theorem 2.1, $\tilde{f}$ is continuous. Then for every $\varepsilon > 0$, there exists $\delta^* > 0$ such that for the oscillation $\omega\left(\tilde{f},[c,d]\right)$ of $\tilde{f}$ over the interval $[c,d]$ we have $\omega\left(\tilde{f},[c,d]\right) < \varepsilon$ if $0 < d-c < \delta^*$ and there is a gauge δ of $[a,b]$ for which $\delta(t) < \delta^*/2$, $t \in [a,b]$, such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_{i=1}^{|d|} \left\| f(\xi_i)(t_i - t_{i-1}) - {}^{K}\!\!\int_{[t_{i-1},t_i]} f(t)\,dt \right\| < \varepsilon.$$

Thus,

$$\sum_{i=1}^{|d|} \left\| \alpha(\xi_i) f(\xi_i)(t_i - t_{i-1}) - {}^{K}\!\!\int_{[t_{i-1},t_i]} \alpha(t) f(t)\,dt \right\| \leq$$

$$\leq \sum_{i=1}^{|d|} \left\| \alpha(\xi_i) \left\{ f(\xi_i)(t_i - t_{i-1}) - {}^{K}\!\!\int_{[t_{i-1},t_i]} f(t)\,dt \right\} \right\| +$$

$$+ \sum_{i=1}^{|d|} \left\| {}^{K}\!\!\int_{[t_{i-1},t_i]} [\alpha(t) - \alpha(\xi_i)] f(t)\,dt \right\| <$$

$$< \|\alpha\|_\infty \varepsilon + \sum_{i=1}^{|d|} \left\| {}^K\!\!\int_{[t_{i-1},t_i]} [\alpha(t) - \alpha(\xi_i)] f(t)\, dt \right\| .$$

But by Corollary 3.3 we have $\alpha f \in K([a,b], Y)$ where

$$ {}^K\!\!\int_{[a,b]} \alpha(t) f(t)\, dt = \int_{[a,b]} \alpha(t)\, d\tilde{f}(t) =$$

$$= \alpha(b) \tilde{f}(b) - \alpha(a) \tilde{f}(a) - \int_{[a,b]} d\alpha(t)\, \tilde{f}(t)$$

and a similar formula holds also for every subinterval contained in $[a,b]$. Hence, for $\beta_{t_i} = [\alpha(t_i) - \alpha(\xi_i)] \tilde{f}(t_i)$ and $\beta_{t_{i-1}} = [\alpha(t_{i-1}) - \alpha(\xi_i)] \tilde{f}(t_{i-1})$ we have

$$\sum_{i=1}^{|d|} \left\| {}^K\!\!\int_{[t_{i-1},t_i]} [\alpha(t) - \alpha(\xi_i)] f(t)\, dt \right\| =$$

$$= \sum_{i=1}^{|d|} \left\| \beta_{t_i} - \beta_{t_{i-1}} - \int_{[t_{i-1},t_i]} d\alpha(t)\, \tilde{f}(t) \right\| =$$

$$= \sum_{i=1}^{|d|} \left\| \beta_{t_i} - \int_{[\xi_i,t_i]} d\alpha(t)\, \tilde{f}(t) - \beta_{t_{i-1}} - \int_{[t_{i-1},\xi_i]} d\alpha(t)\, \tilde{f}(t) \right\| =$$

$$= \sum_{i=1}^{|d|} \left\| \int_{[\xi_i,t_i]} d\alpha(t) \left[\tilde{f}(t_i) - \tilde{f}(t) \right] + \int_{[t_{i-1},\xi_i]} d\alpha(t) \left[\tilde{f}(t_{i-1}) - \tilde{f}(t) \right] \right\| \leq$$

$$\leq V(\alpha)\, \varepsilon,$$

because $\left\| \tilde{f}(t_i) - \tilde{f}(t) \right\| \leq \omega\left(\tilde{f}, [t_{i-1}, t_i] \right)$ and $\left\| \tilde{f}(t_{i-1}) - \tilde{f}(t) \right\| \leq \omega\left(\tilde{f}, [t_{i-1}, t_i] \right)$ for every $t \in [t_{i-1}, t_1]$. The proof is complete. □

The next result can be proved by appeal to the definitions.

Theorem 3.4 *Let $\alpha \in H([a,b], L(X,Y))$ and $f \in K^{\tilde{\alpha}}([a,b], X)$ (respectively $f \in H^{\tilde{\alpha}}([a,b], X)$). If f is bounded, then $\alpha f \in K([a,b], Y)$ (respectively $\alpha f \in H([a,b], Y)$) and*

$$ {}^K\!\!\int_{[a,b]} \alpha(t) f(t)\, dt = {}^K\!\!\int_{[a,b]} d\tilde{\alpha}(t)\, f(t) .$$

Corollary 3.4 *Suppose* $\alpha \in H([a,b], L(X,Y))$ *with* $\tilde{\alpha} \in SV([a,b], L(X,Y))$ *and* $f \in G([a,b], X)$. *Then* $\alpha f \in K([a,b], Y)$ *and*

$$^{K}\int_{[a,b]} \alpha(t) f(t)\, dt = {}^{K}\int_{[a,b]} d\tilde{\alpha}(t)\ f(t).$$

P r o o f. By Theorem 2.1, $\tilde{\alpha} \in C([a,b], L(X,Y))$. Then the result comes by Theorem 2.3, since $C([a,b], L(X,Y)) \subset G^{\sigma}([a,b], L(X,Y))$. $\square$

Corollary 3.5 *Suppose* $\alpha \in H([a,b], L(X,Y))$ *with* $\tilde{\alpha} \in SV([a,b], L(X,Y))$ *and* $f \in C([a,b], X)$. *Then* $\alpha\, f \in K([a,b], Y)$ *and we have*

$$^{K}\int_{[a,b]} \alpha(t) f(t)\, dt = \int_{[a,b]} d\tilde{\alpha}(t)\ f(t) \tag{3}$$

and

$$\int_{[a,b]} d\tilde{\alpha}(t)\ f(t) = \tilde{\alpha}(b) f(b) - \tilde{\alpha}(a) f(a) - \int_{[a,b]} \tilde{\alpha}(t)\ df(t). \tag{4}$$

Corollary 3.6 *If* $\alpha \in H([a,b], L(X,Y))$ *and* $f \in BV([a,b], X)$, *then* $\alpha f \in K([a,b], Y)$ *and equations (3) and (4) hold.*

P r o o f. Follows by Theorems 2.1 and 2.2. $\square$

More generally we have

Theorem 3.5 *If* $\alpha \in K([a,b], L(X,Y))$ *and* $f \in BV([a,b], X)$, *then* $\alpha f \in K([a,b], Y)$ *and both (3) and (4) hold.*

Theorem 3.6 *If* $\alpha \in H([a,b], L(X,Y))$ *and* $f \in BV([a,b], X)$, *then* $\alpha f \in H([a,b], Y)$ *and equations (3) and (4) hold.*

The proof of Theorem 3.5 follows the steps of the proof of Theorem 4.2 below applying Theorems 2.1 and 2.2. The proof of Theorem 3.6 is analogous to that of Theorem 3.3.

4 The Substitution Formulas

In this section, beside X and Y, W also denotes a Banach space.

4.1 Substitution Formulas for the Kurzweil vector integrals

Theorem 4.1 *Let* $f : [a,b] \to X$, $\alpha \in K_f([a,b], L(X,W))$, $g = \tilde{\alpha}_f : [a,b] \to W$ *(i.e.,* $g(t) = {}^K\int_{[a,t]} \alpha(s)\, df(s)$, *for every* $t \in [a,b]$*) and* $\gamma \in SV([a,b], L(W,Y))$. *Then* $\gamma \in K_g([a,b], L(W,Y))$ *if and only if* $\gamma\,\alpha \in K_f([a,b], L(X,Y))$. *In this case,*

$$ {}^K\int_{[a,b]} \gamma(t)\, \alpha(t)\ df(t) = {}^K\int_{[a,b]} \gamma(t)\ dg(t). \tag{5}$$

P r o o f. Since $\alpha \in K_f([a,b], L(X,W))$, then given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\| \sum_{i=1}^{|d|} \left\{ \alpha(\xi_i)\left[f(t_i) - f(t_{i-1})\right] - {}^K\int_{[t_{i-1},t_i]} \alpha(t)\, df(t) \right\} \right\| < \varepsilon.$$

Taking approximated sums for ${}^K\int_{[a,b]} \gamma(t)\, \alpha(t)\, df(t)$ and ${}^K\int_{[a,b]} \gamma(t)\, dg(t)$ we have

$$\left\| \sum_{i=1}^{|d|} \gamma(\xi_i)\, \alpha(\xi_i)\left[f(t_i) - f(t_{i-1})\right] - \sum_{i=1}^{|d|} \gamma(\xi_i)\left[g(t_i) - g(t_{i-1})\right] \right\|$$

$$= \left\| \sum_{i=1}^{|d|} \gamma(\xi_i) \left\{ \alpha(\xi_i)\left[f(t_i) - f(t_{i-1})\right] - {}^K\int_{[t_{i-1},t_i]} \alpha(t)\, df(t) \right\} \right\| = I.$$

However, if $\gamma_i \in L(X,Y)$ and $x_i \in X$, then

$$\sum_{i=i}^{n} \gamma_i x_i = \sum_{j=1}^{n} (\gamma_j - \gamma_{j-1}) \left(\sum_{i=j}^{n} x_i \right) + \gamma_0 \left(\sum_{i=j}^{n} x_i \right),\ n \in \mathbb{N}.$$

Thus, for $x_i = \alpha(\xi_i)\left[f(t_i) - f(t_{i-1})\right] - {}^K\int_{[t_{i-1},t_i]} \alpha(t)\, df(t)$, $\gamma_i = \gamma(\xi_i)$, $\gamma_0 = \gamma(a)$ and $n = |d|$ we have

$$I = \left\| \sum_{j=1}^{|d|} \left[\gamma(\xi_j) - \gamma(\xi_{j-1})\right] \left(\sum_{i=j}^{|d|} x_i \right) + \gamma_0 \left(\sum_{i=j}^{|d|} x_i \right) \right\| \le$$

$$< SV(\gamma)\, \varepsilon + \|\gamma(a)\|\, \varepsilon,$$

once $\left\| \sum_{i=j}^{|d|} x_i \right\| \le \varepsilon$ for every $j \in \{1, 2, \ldots, |d|\}$, by the Saks-Henstock Lemma (Lemma 2.1). $\square$

Corollary 4.1 *Let* $f : [a, b] \to X$, $\alpha \in K_f([a, b], L(X, W))$, $g = \tilde{\alpha}_f \in BV([a, b], W)$ *and* $\gamma \in G([a, b], L(W, Y))$. *Then* $\gamma \in K_g([a, b], L(W, Y))$, $\gamma\,\alpha \in K_f([a, b], L(X, Y))$ *and equation (5) holds.*

P r o o f. By Theorem 2.4, $\gamma \in K_g([a, b], L(W, Y))$. $\square$

Corollary 4.2 *If* $f \in C([a, b], X)$, $\alpha \in K_f([a, b], L(X, W))$, $g = \tilde{\alpha}_f : [a, b] \to W$ *and* $\gamma \in SV([a, b], L(W, Y))$, *then* $\gamma \in R_g([a, b], L(W, Y))$, $\gamma\,\alpha \in K_f([a, b], L(X, Y))$ *and equation (5) holds.*

P r o o f. $\gamma \in R_g([a, b], L(W, Y)) \subset K_g([a, b], L(W, Y))$, by Theorems 2.1 and 2.2. $\square$

Theorem 4.2 *Let* $\gamma : [a, b] \to L(W, Y)$, $\alpha \in K^\gamma([a, b], L(X, W))$, $f \in BV([a, b], X)$ *and* $\beta = \tilde{\alpha}^\gamma : [a, b] \to L(X, Y)$ *(i.e.,* $\beta(t) = {}^K\int_{[a,t]} d\gamma(s)\,\alpha(s)$, *for every* $t \in [a, b]$*). Then* $f \in K^\beta([a, b], X)$ *if and only if* $\alpha\, f \in K^\gamma([a, b], Y)$. *In this case,*

$$ {}^K\int_{[a,b]} d\gamma(t)\ \alpha(t)\, f(t) = {}^K\int_{[a,b]} d\beta(t)\ f(t)\,. \tag{6} $$

P r o o f. By hypothesis, $\alpha \in K^\gamma([a, b], L(X, W))$. Hence, given $\varepsilon > 0$, there exists a gauge δ of $[a, b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$ \left\| \sum_{i=1}^{|d|} \left\{ [\gamma(t_i) - \gamma(t_{i-1})]\, \alpha(\xi_i) - {}^K\int_{[t_{i-1},t_i]} d\gamma(t)\, \alpha(t) \right\} \right\| < \varepsilon. $$

Taking approximated sums for ${}^K\int_{[a,b]} d\gamma(t)\,\alpha(t)\,f(t)$ and ${}^K\int_{[a,b]} d\beta(t)\,f(t)$ we have

$$ \left\| \sum_{i=1}^{|d|} [\gamma(t_i) - \gamma(t_{i-1})]\, \alpha(\xi_i)\, f(\xi_i) - \sum_{i=1}^{|d|} [\beta(t_i) - \beta(t_{i-1})]\, f(\xi_i) \right\| = $$

$$ = \left\| \sum_{i=1}^{|d|} \left\{ [\gamma(t_i) - \gamma(t_{i-1})]\, \alpha(\xi_i) - {}^K\int_{[t_{i-1},t_i]} d\gamma(t)\, \alpha(t) \right\} f(\xi_i) \right\| $$

But, if $\gamma_i \in L(X, Y)$ and $x_i \in X$, then

$$ \sum_{i=1}^{n} \gamma_i x_i = \left(\sum_{i=1}^{n} \gamma_i \right) x_0 + \sum_{j=1}^{n} \left(\sum_{i=j}^{n} \gamma_i \right) (x_j - x_{j-1})\,,\ n \in \mathbb{N}. $$

Hence, taking $\gamma_i = [\gamma(t_i) - \gamma(t_{i-1})]\,\alpha(\xi_i) - {}^K\int_{[t_{i-1},t_i]} d\gamma(t)\,\alpha(t)$, $x_i = f(\xi_i)$, $x_0 = f(a)$ and $n = |d|$, we obtain

$$I \leq \left\| \sum_{i=1}^{|d|} \left\{ [\gamma(t_i) - \gamma(t_{i-1})]\,\alpha(\xi_i) - {}^K\int_{[t_{i-1},t_i]} d\gamma(t)\,\alpha(t) \right\} \right\| \|f(a)\| +$$

$$+ \sum_{j=1}^{|d|} \left\| \sum_{i=j}^{|d|} \gamma_i \right\| \|f(t_i) - f(t_{i-1})\| < \varepsilon \|f(a)\| + \varepsilon V(f),$$

since $\left\| \sum_{i=j}^{|d|} \gamma_i \right\| = \left\| \sum_{i=j}^{|d|} \left\{ [\gamma(t_i) - \gamma(t_{i-1})]\,\alpha(\xi_i) - {}^K\int_{[t_{i-1},t_i]} d\gamma(t)\,\alpha(t) \right\} \right\| \leq \varepsilon$, for every $j \in \{1, 2, \ldots, |d|\}$, by the Saks-Henstock Lemma (Lemma 2.1). □

Corollary 4.3 *Suppose* $\gamma \in G^\sigma([a,b], L(W,Y))$, $\alpha \in K^\gamma([a,b], L(X,W))$, $\beta = \tilde{\alpha}^\gamma \in SV([a,b], L(X,Y))$ *and* $f \in G([a,b], X)$. *Then* $f \in K^\beta([a,b], X)$, $\alpha f \in K^\gamma([a,b], Y)$ *and equation (6) holds.*

P r o o f. By Theorem 2.1, $\beta \in G([a,b], L(X,Y))$. Then Theorem 2.3 implies that $f \in K^\beta([a,b], X)$. □

Corollary 4.4 *If* $\gamma \in C^\sigma([a,b], L(W,Y))$, $\alpha \in K^\gamma([a,b], L(X,W))$, $f \in BV([a,b], X)$ *and* $\beta = \tilde{\alpha}^\gamma : [a,b] \to L(X,Y)$, *then* $f \in R^\beta([a,b], X)$, $\alpha f \in K^\gamma([a,b], Y)$ *and equation (6) holds.*

P r o o f. $f \in R^\beta([a,b], X)$ by Theorems 2.1 and 2.2, and $R^\beta([a,b], X) \subset K^\beta([a,b], X)$. □

Remark 4.1 *Similar results are valid for the Riemann-Stieltjes integrals, since the Saks-Henstock Lemma holds for* $R^\alpha([a,b], X)$ *and* $R_f([a,b], L(X,Y))$ *instead of* $K^\alpha([a,b], X)$ *and* $K_f([a,b], L(X,Y))$ *respectively.*

4.2 Substitution Formulas for the Henstock vector integrals

The results of this subsection are straight applications of the definitions.

Theorem 4.3 *Let* $f : [a,b] \to X$, $\alpha \in H_f([a,b], L(X,W))$, $g = \tilde{\alpha}_f : [a,b] \to W$ *(i.e.,* $g(t) = {}^K\int_{[a,t]} \alpha(s)\, df(s)$, *for every* $t \in [a,b]$*) and let* $\gamma : [a,b] \to L(W,Y)$ *be bounded. Then* $\gamma \in H_g([a,b], L(W,Y))$ *if and only if* $\gamma\alpha \in H_f([a,b], L(X,Y))$. *In this case, equation (5) holds.*

Corollary 4.5 *Let $f \in H([a,b], X)$. If $\alpha : [a,b] \to L(X,Y)$ is bounded, then $\alpha \in H_{\tilde{f}}([a,b], L(X,Y))$ if and only if $\alpha f \in H([a,b], Y)$. In this case,*

$$^{K}\int_{[a,b]} \alpha(t)\, f(t)\, dt = {}^{K}\int_{[a,b]} \alpha(t)\; d\tilde{f}(t)\,.$$

Theorem 4.4 *Let $\gamma : [a,b] \to L(W,Y)$, $\alpha \in H^{\gamma}([a,b], L(X,W))$, $\beta = \tilde{\alpha}^{\gamma} : [a,b] \to L(X,Y)$ (i.e., $\beta(t) = {}^{K}\int_{[a,t]} d\gamma(s)\; \alpha(s)$, for every $t \in [a,b]$) and let $f : [a,b] \to X$ be bounded. Then $f \in H^{\beta}([a,b], X)$ if and only if $\alpha f \in H^{\gamma}([a,b], Y)$. In this case, equation 6 holds.*

Corollary 4.6 *Let $\alpha \in H([a,b], L(X,Y))$ and $f : [a,b] \to X$ be bounded. Then $f \in H^{\tilde{\alpha}}([a,b], X)$ if and only if $\alpha f \in H([a,b], Y)$. In this case,*

$$^{K}\int_{[a,b]} \alpha(t)\, f(t)\, dt = {}^{K}\int_{[a,b]} d\tilde{\alpha}(t)\; f(t)\,.$$

Remark 4.2 *Similar results hold for the vector integrals of McShane which gives a Riemannian definition of the Bochner-Lebesgue-Stieltjes integrals when we introduce McShane's modification in the Henstock vector integrals ([8], [6]).*

Acknowledgement. We thank Prof. Dr. Chaim Samuel Hönig for the suggestion of the subject.

References

[1] *Federson, M.:* The Fundamental Theorem of Calculus for multidimensional Banach space-valued Henstock vector integrals, Real Analysis Exchange, 25, 1, (2000), 469-480.

[2] *Federson, M. and Bianconi, R.:* Linear integral equations of Volterra concerning the integral of Henstock, Real Anal. Exchange, 25, 1, (2000), 389-417.

[3] *Henstock, R.:* The General Theory of Integration, Oxford Math. Monog., Clarendon Press, Oxford, 1991.

[4] *Hönig, C. S.:* Volterra-Stieltjes integral equations, Math. Studies, 16, North-Holland Publ. Comp., Amsterdam, 1975.

[5] *Hönig, C. S.:* There is no natural Banach space norm on the space of Kurzweil-Henstock-Denjoy-Perron integrable functions, Seminrio Brasileiro de Anlise, 30 (1989), 387-397.

[6] *Hönig, C. S.:* A Riemaniann characterization of the Bochner-Lebesgue integral, Seminrio Brasileiro de Anlise, 35 (1992), 351-358.

[7] *Lee, P. Y.:* Lanzhou Lectures on Henstock Integration, World Sci., Singapore, 1989.

[8] *McShane, E. J.:* A unified theory of integration, Am. Math. Monthly, 80 (1973), 349-359.

[9] *Schwabik, S.:* Abstract Perron-Stieltjes integral, Math. Bohem., 121 (1996), 4, 425-447.

Author's address: Universidade de São Paulo, Departamento de Matemática - ICMC, CP 688 São Carlos - SP 13560-000, Brazil. E-mail: `federson@icmc.sc.usp.br`