Simulating Viscoplastic Fluid using Smoothed Particle Hydrodinamics

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ABSTRACT

The main purpose of this work is to simulate viscoplastic properties of materials such as metal, plastic, wax, polymer and lava. The technique consists in modeling the object as a non-Newtonian fluid with high viscosity. The viscosity is formulated using the General Newtonian fluid model and fluid simulation framework uses a variation of the Lagrangian method called Smoothed Particle Hydrodynamics. This work also includes several schemes that improve the efficiency and the numerical stability of the equations. We show some applications in melting objects and debris flow.

INTRODUCTION

Among non-Newtonian fluids, viscoplastic fluids are characterized by the effect that a significant force must be applied to them before it starts to flow. The critical of the external force is known as the *yield stress*.

The recent advances of Mendes et al. [1] formulate the viscosity with a general Newtonian law which encompasses both viscous and liquid state. The conciseness and generality of this formulation suits better for a lot of applications like melting for example.

Simulating the fluid behavior of a viscoplastic object requires a computational fluid dynamics (CFD) framework. In the rheology literature, the most common CFD model relies on Eulerian formulation where physical quantities are sampled on a regular grid. This suits well for classical Newtonian fluids like water. However, controlling grid based methods require tracking the boundary of the fluid, which remains a laborious task in free flow simulations.

In this work, we use a Lagrangian formulation on a particle-based representation, called *Smoothed Particle Hydrodynamics* (SPH). The SPH method was introduced in 1977 by Gingold and Monaghan [2] and Lucy [3] simulate compressible fluids in astrophysics. Each particle represents a small volume of fluid subjected to natural forces such as gravity, pressure and viscosity. SPH methods are simple to implement and its accuracy compares nicely to grid based methods in several instances.

METHODOLOGY

Put body of the paper here. Put body of the paper Computational fluid dynamics (CFD) aims at prediction fluid behavior through Navier-Stokes equations. These equations are commonly solved using conventional Eulerian formulation with grid-based methods such as finite differences and finite elements. In this work, we chose an alternative method driven by the Lagrangian formulation.

As opposed to the Eulerian approach, the Lagrangian formulation does not require advective term, which suits well for meshless methods such as SPH. We will introduce now this formulation and the General Newtonian Fluid model [1] that models our viscoplastic object.

Lagrangian formulation of Navier-Stokes equations can be formulated by the following two equations describing the conservation of the mass (equation (1.1)) and of the momentum (equation (1.2)).

$$\frac{d\rho}{dt} = -\rho \nabla . \mathrm{v} \tag{1.1}$$

$$\frac{d\mathrm{v}}{dt} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla . \mathrm{S} + \mathrm{g} \tag{1.2}$$

where t denotes the time, v the velocity vector, the density, p the pressure, g the gravity acceleration vector and S the viscoplastic stress tensor.

Generalized Newtonian Fluid model. For non-Newtonian fluids, the stress tensor is a nonlinear function of the deformation tensor $\mathrm{D} = \nabla \mathrm{v} + \left(\nabla \mathrm{v}\right)^{\mathrm{T}}$. For our simulation, we will use the Generalized Newtonian Liquid model proposed by Mendes et al. [1], where the stress tensor S is given by $\mathrm{S} = \eta\left(D\right)\mathrm{D}$, where the apparent viscosity η depends on the intensity of deformation $D = \sqrt{\frac{1}{2}.\mathrm{tr}\left(\mathrm{D}\right)^2}$. The viscosity function η is then given by:

$$\eta\left(D\right) = \left(1 - \exp\left[-\left(J+1\right)D\right]\right)\left(D^{n-1} + \frac{1}{D}\right) \tag{1.3}$$

where n is the behavior of power-law index and J is the *jump number*.

The jump number J is a new rheological parameter of a viscoplastic fluid which combines previous ones such as the yield stress and the consistency index. In our simulations, we fixed n = 0.5.

Smoothed Particle Hydrodynamics (SPH). The key idea of SPH is to replace the fluid by a set of particles.

The dynamics of the fluid is then naturally governed by the Lagrangian version of the Navier-Stokes equations (equations (1.1) and (1.2)). The local fluid properties such as mass and volume are attached to each particle and interpolated in-between particles. This interpolation uses a smoothing kernel *W* on the particles in a radius of *h*: A scalar field *A*(x) and its associated gradient vector field *A*(x) at point x are interpolated using the particles *j* within a disk of radius *h* around x as follows:

$$A(x)=\sum_{j=1}^{n} A(\mathrm{x}_j)\frac{m_j}{\rho_j}W(\mathrm{x}-\mathrm{x}_j,h)$$

$$\nabla A(x)=\sum_{j=1}^{n} A(\mathrm{x}_j)\frac{m_j}{\rho_j}\nabla W(\mathrm{x}-\mathrm{x}_j,h)$$

where n is the number of neighboring particles, j the particle index, x_j the particle position, m_j the particle mass and ρ_j the particle density.

In this work, we choose a piecewise sextic smoothing kernel function,

$$W(\mathrm{r})=\begin{cases}(1-\mathrm{r}^2)^3, & \text{if } \|\mathrm{r}\|\le 1\\ 0, & \text{otherwise}\end{cases}$$

where $\mathrm{r}=\dfrac{\|\mathrm{x}-\mathrm{y}\|}{h}$.

Particle approximation of continuity. We chose symmetrization for the density, using the following SPH version of continuity equation (1.1):

$$\frac{d\rho_i}{dt}=\rho_i\sum_{j=1}^{n}\frac{m_j}{\rho_j}(\mathrm{v}_i-\mathrm{v}_j).\nabla_i W(\mathrm{x}_{ij},h)$$

where v_i and v_j are velocities at particles i and j respectively, and $\mathrm{x}_{ij}=\mathrm{x}_i-\mathrm{x}_j$.

Particle approximation of the momentum. The modeling of pressure remains a delicate point for SPH simulations of incompressible fluids, due to the lack of explicit control of the local density. Since SPH suits better for compressible fluid, we approximate the incompressible fluid by a quasi-compressible fluid through an equation of state [4] for the pressure. We use the one proposed by Morris et al. [5]:

$$p_i=c^2(\rho_i-\rho_0) \tag{1.4}$$

where p_i is the pressure at particle i, c the speed of sound, which represents the fastest velocity of a wave propagation in that medium, and ρ_0 is a reference density.

After updating the pressure at all particles using equation (1.4), we can evaluate the pressure term in equation (1.2) at each particle i using a symmetrization similar to the density case [6]:

$$-\frac{1}{\rho_i}\nabla p_i=-\sum_{j=1}^{n} m_j\left(\frac{p_i}{\rho^2{}_i}+\frac{p_j}{\rho^2{}_j}\right)\nabla_i W(\mathrm{x}_{ij},h).$$

Stress tensor. In order to compute the stress tensor $\mathrm{S}_i=\eta(D_i)\mathrm{D}_i$ at each particle i, where $\eta(D_i)$ is given by the equation (1.3) we must pre-compute the deformation tensor:

$$\mathrm{D_i}=\nabla\mathrm{v}_i+(\nabla\mathrm{v}_i)^{\mathrm{T}}$$

where the velocity is evaluated by the following equation:

$$\nabla\mathrm{v}_i=\sum_{j=1}^{n}\frac{m_j}{\rho_j}(\mathrm{v}_j-\mathrm{v}_i)\otimes\nabla_i W(\mathrm{x}_{ij},h).$$

Finally, after updating of the stress tensor S_i at each particle i, the stress term in equation (1.2) can be approximated by:

$$\frac{1}{\rho_i}\nabla.S_i=\sum_{j=1}^{n}\frac{m_j}{\rho_i\rho_j}(\mathrm{S}_i+\mathrm{S}_j).\nabla_i W(\mathrm{x}_{ij},h).$$

A wide review of SPH method can be found in Monaghan's survey [7].

ACKNOWLEDGMENTS

We would like to thank Prof. Paulo Roberto Mendes (Department of Mechanical Engineering, PUC-Rio) for suggesting us to use his viscoplastic fluid model. The authors are members of Matmidia laboratory at PUC-Rio which is sponsored by CNPq, FAPERJ and Petrobras.

REFERENCES

[1] P. R. S. Mendes, E. S. S. Dutra, J. R. R. Siffert, and M. F. Naccache. Gas displacement of viscoplastic liquids in cappilary tubes. *Journal of Non-Newtonian Fluid Mechanics*, 2005. (to appear).

[2] R. A. Gingold and J. J. Monaghan. Smoothed particle hydrodynamics: theory and application to non-spherical stars. Monthly *Notices of the Royal Astronomical Society*, 181:375-389, 1977.

[3] L. B. Lucy. Numerical approach to testing the fission hyphotesis. *Astronomical Journal*, 82:1013-1024, 1977.

[4] J. J. Monaghan. Simulating free surface flow with sph. *Journal of Computational Physics*, 110:399-406, 1994.

[5] J. P. Morris, P. J. Fox, and Y. Zhu. Modeling low reynolds number for incompressible flows using SPH. *Journal of Computational Physics*, 136:214-226, 1997.

[6] S. Li and W. K. Liu. *Meshfree Particle Methods*. Springer, 2004.

[7] J. J. Monaghan. Smoothed particle hydrodynamics. *Reports on Progress in Physics*, 68:1703-1759, 2005.